Lecture 26 (6/3/2019)

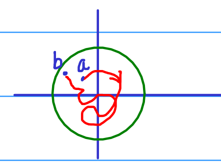
Within the region of convergence, a power series define a holomorphic function $f(z) = a_0 + a_1 z + a_2 z^2 + \dots$ for all $z \in D_R(0)$.

One can differentiate term by term:

$$f'(z) = 0 + a_1 + 2a_2 z + 3a_3 z^2 + \dots \quad \forall z \in D_R(0).$$

One can also integrate term by term provided that $\gamma \subset D_R(0)$:

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{\infty} a_k \int_\gamma z^k\,dz = \sum_{k=0}^{\infty} a_k \left.\frac{z^{k+1}}{k+1}\right|_b^a = \sum_{k=0}^{\infty} a_k \frac{b^{k+1} - a^{k+1}}{k+1}$$



Def: Let f be a function of complex variable and $z_0 \in \mathbb{C}$. If f can be represented as $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$ for $z \in D_r(z_0)$ (some small disk centered at z_0) then f is said to be analytic at z_0.

Note that a power series representation of f, if exists, is unque and is called Taylor series of f. More specifically, each coefficient a_k is uniquely defined by f:

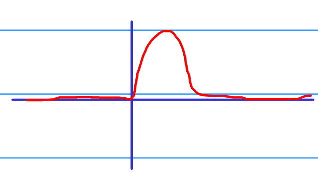
$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

We have seen that if f is analytic (at z_0) then f is holomorphic at z_0. How about the converse?

$$\text{holomorphic} \overset{?}{\Rightarrow} \text{analytic}$$

The answer is NO for real variables. For example, function

$$f(x) = \begin{cases} e^{\frac{1}{x(x-1)}} & \text{if } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$



is infinitely differentiable with $f^{(k)}(0) = 0$ for all k.

$$f(x) \neq \underbrace{a_0}_{0} + \underbrace{a_1}_{0} x + \underbrace{a_2}_{0} x^2 + \underbrace{a_3}_{0} x^3 + \dots$$

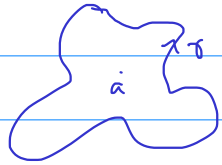
However, this phenominon doesn't occur in Complex variables. Most phenomena that are unique to complex variables can trace back to Cauchy's Integral formula. This formula shows a strange phenomenon of Complex variables.

The only simple loop in $\mathbb{R}$ is a point. The integral over any simple loop is equal to zero.

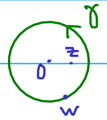
$$\int_a^a f(x)\,dx = 0$$

However, it is not the case for complex variabled functions:

$$\int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a) \neq 0$$



Let f be a holomorphic function at 0. Then f is holomorphic on a small disk $D_r(0)$. We will show that f is equal to a power series on this disk.



$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\,dz \qquad \forall z \in D_r(0).$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w} \frac{1}{1-\frac{z}{w}}\,dw$$

Now that $\left|\frac{z}{w}\right| = \frac{|z|}{r} < 1$

we can write

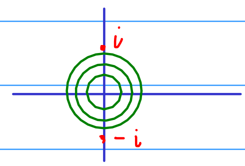
$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w} \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \frac{z^3}{w^3} + \dots\right) dw$$

$$= \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w}\,dw}_{a_0} + \underbrace{\left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^2}\,dw\right)}_{a_1} z + \underbrace{\left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^3}\,dw\right)}_{a_2} z^2 + \dots$$

Therefore, f is analytic (at some point z_0) if and only if f is holomorphic at z_0. Analyticity and holomorphicity are equivalent.

* Principle: f fails to be analytic if and only if it fails to be holomorphic.

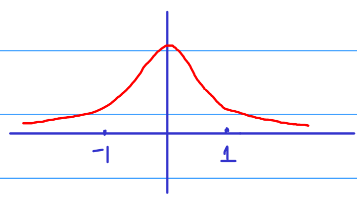
Ex:
$$f(z) = \frac{1}{1+z^2} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$



Without writing what each a_k is, one can tell that this power series has radius of convergence equal to 1. This is the distance from 0 to the closest point where f fails to be holomorphic.

The explicit form
$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = 1 - z^2 + z^4 - z^6 + \dots$$
confirm this observation.

In comparison with real variabled function $f(x) = \dfrac{1}{1+x^2}$:

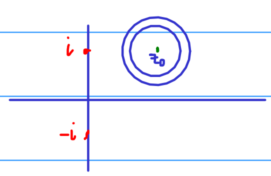


it is not obvious (without any computation) why the Taylor series of this function should cease to converge as $|x| > 1$.

In complex plane, we see that this is because the function $\frac{1}{1+z^2}$ ceases to be holomorphic as $|z| \to 1^-$.

Ex:
$$f(z) = \frac{1}{1+z^2} = a_0 + a_1(z-1-i) + a_2(z-1-i)^2 + \dots$$

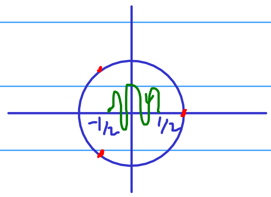
What is the radius of convergence of this series?



It is equal to the shorter of the distances from $z_0 = 1+i$ to i and to $-i$, which is 1.

Ex :

$$f(z) = \frac{1}{z^3-1}$$



γ is a path from $1/2$ to $-1/2$ as in the picture. Find $\int_\gamma f(z)\,dz$.

We see that f is holomorphic everywhere except for three points, which are three roots of $z^3 = 1$. Inside the unit circle, f is holomorphic. Thus, it is analytic at 0, which region of convergence $D_1(0)$.

$$f(z) = \frac{-1}{1-z^3} = -1 - z^3 - z^6 - \cdots = -\sum_{k=0}^{\infty} z^{3k}$$

Since γ lies inside $D_1(0)$, one can integrate term by term:

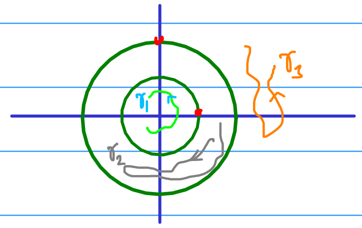
$$\int_\gamma f(z)\,dz = -\sum_{k=0}^{\infty} \int_\gamma z^{3k}\,dz = -\sum_{k=0}^{\infty} \left.\frac{z^{3k+1}}{3k+1}\right|_{1/2}^{-1/2}$$

$$= -\sum_{k=0}^{\infty} \frac{(-1/2)^{3k+1} - (\frac{1}{2})^{3k+1}}{3k+1}$$

$\approx \ldots$ (truncate to approximate)

Ex :

$$f(z) = \frac{1}{(z-1)(z-2i)}$$



How to compute $\underbrace{\int_{\gamma_1} f(z)\,dz}_{\text{already discussed}}$, $\int_{\gamma_2} f(z)\,dz$, $\int_{\gamma_3} f(z)\,dz$?

One needs a generalization of power series, called Laurent series. This is a type of power series in which the powers are allowed to be negative.

$$\cdots a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k=-\infty}^{\infty} a_k z^k$$

How to interpret this series?

$$\cdots a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} a_{-k} z^{-k}$$

sum neg. powers / \ sum positive powers