Lecture 27 (6/5/2019)

If f is holomorphic at z_0 then

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \qquad \forall z \text{ near } z_0.$$

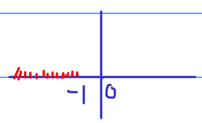
Consequently,

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad \text{for any } z \in \mathbb{C}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \qquad \text{for any } z \in \mathbb{C}$$

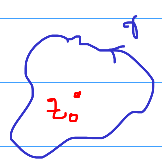
$$\mathrm{Log}(z+1) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \qquad \text{for any } z \in D_1(0).$$

e^z, $\cos z$, $\sin z$ are holomorphic everywhere (entire functions). Thus the region of convergence is $\mathbb{C}$. Radius of convergence is ∞.



$\mathrm{Log}(z+1)$ is not holomorphic on $\mathbb{R}_{\leq -1}$. Radius of convergence is 1.

<u>Question</u>: if f is not holomorphic at z_0 (i.e. z_0 is a singular point or singularity of f), can f be still written as some type of series?



In other words, if f is not holomorphic at z_0, and that a loop γ encloses z_0, we still wish to compute

$$\int_\gamma f(z)\,dz$$

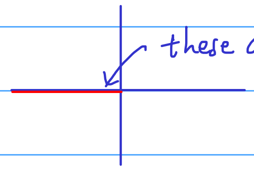
by "mode decomposition" method.

<u>Answer</u>:

If z_0 is an isolated singularity of f then can be written as a $\underbrace{\text{Laurent series}}$ around z_0.

generalized Taylor series where the powers are allowed to be negative.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$



these are non-isolated singularities

There are only 3 types of isolated singularities:

- Removable singularity: not a singularity (no negative powers)

  For example $z=0$ is a removable sing. of $\frac{\sin z}{z}$ because

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

- Pole: finitely many negative powers. For example,

$$z^{-3} e^{z} = z^{-3} + \frac{1}{1!} z^{-2} + \frac{1}{2!} z^{-1} + \frac{1}{3!} + \frac{1}{4!} z + \cdots \qquad \text{for } z \in \mathbb{C} \setminus \{0\}$$

  $z=0$ is a pole of order 3.

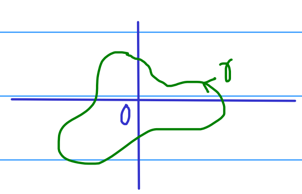
- Essential singularity: infinitely many negative powers. For example,

$$e^{1/z} = \cdots + \frac{1}{2!} z^{-2} + \frac{1}{1!} z^{-1} + 1 \qquad \text{for } z \in \mathbb{C} \setminus \{0\}$$

$$z^3 e^{1/z} = \cdots + \frac{1}{4!} z^{-1} + \frac{1}{3!} + \frac{1}{2!} z + \frac{1}{1!} z^2 + z^3 \qquad \text{for } z \in \mathbb{C} \setminus \{0\}$$

In both cases, $z=0$ is an essential singularity.

Ex: $\displaystyle\int_\gamma (z+1) e^{-1/z} \, dz = ?$



$$(z+1) e^{-1/z} = (z+1)\left(1 - \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} - \frac{1}{6}\frac{1}{z^3} + \cdots\right)$$

$$= \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 + a_1 z + \cdots$$

$a_0 = -1 + 1 = 0$

$a_{-1} = \frac{1}{2} - 1 = -\frac{1}{2}$

$\cdots\cdots$

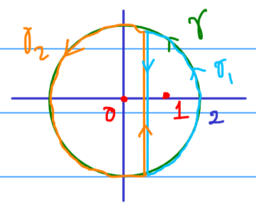
Formally integrate both sides (assuming integral on RHS can be taken term by term):

$$\int_\gamma f(z)\,dz = \cdots + a_{-2}\underbrace{\int_\gamma z^{-2}\,dz}_{0} + a_{-1}\underbrace{\int_\gamma z^{-1}\,dz}_{2\pi i} + a_0\underbrace{\int_\gamma dz}_{0} + a_1\underbrace{\int_\gamma z\,dz}_{0} + \cdots$$

$$= a_{-1}\, 2\pi i$$

We see that a_{-1} is the only coefficient in the Laurent series that matters. It is called the residue of f at 0, denoted by $\operatorname{Res}[f;0]$. The textbook uses notation $\operatorname{Res}_{z=0} f$.

Ex: $\displaystyle\int_\gamma \frac{e^z}{z^3(z-1)} = ?$



γ is the circle of radius 2.

The integrand has two isolated singular points: 0 and 1.

What we used to do is the separate them:

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$$

Each γ_1 and γ_2 encloses only one singularity. Then we use Cauchy's integral formula:

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_1} \frac{z^{-3}e^z}{z-1}\,dz = 2\pi i\left(z^{-3}e^z\right)\Big|_{z=1} = \cdots$$

$$\int_{\gamma_2} f(z)\,dz = \int_{\gamma_2} \frac{e^z/(z-1)}{z^3}\,dz = \cdots$$ (use general Cauchy's formula)

Now there is another way to interpret the above method. Let's write the Laurent series of f about $z=0$ and about $z=1$:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-1)^k \implies \int_{\gamma_1} f(z)\,dz = 2\pi i\, a_{-1} = 2\pi i \operatorname{Res}[f;1]$$

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k \implies \int_{\gamma_2} f(z)\,dz = 2\pi i\, b_{-1} = 2\pi i \operatorname{Res}[f;0]$$

Therefore,
$$\int_{\gamma} f(z)\,dz = 2\pi i \left(\operatorname{Res}[f;1] + \operatorname{Res}[f;0] \right)$$

Theorem (Cauchy's Residue theorem)

Let γ be a simple loop, positively oriented. Let f be a complex function. Suppose $z_1, z_2, \dots, z_m$ are all isolated singularities of f enclosed in γ. Then

$$\int_{\gamma} f(z)\,dz = 2\pi i \left(\operatorname{Res}[f;z_1] + \operatorname{Res}[f;z_2] + \cdots + \operatorname{Res}[f;z_m] \right)$$