Lecture 28 (6/7/2019)

Question: in what case can we write Laurent series?

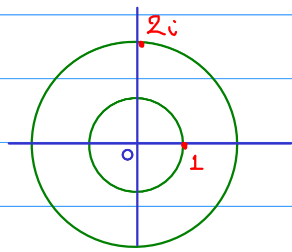
Thm Let f be holomorphic on an annulus $\{z : r < |z| < R\}$, where $0 \le r < R \le \infty$. Then f has a Laurent series representation about 0:

$$f(z) = \cdots + a_{-2} z^{-2} + a_1 z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots$$

Ex:
$$f(z) = \frac{1}{(z-1)(z-2i)}$$

Find the Laurent series representations of f about 0.

The singularities of $f(z)$ are 1 and $2i$. They partition the complex plane into three zones:



1) $|z| < 1$

2) $1 < |z| < 2$

3) $|z| > 2$

By partial fraction,
$$f(z) = \frac{1}{1-2i}\,\frac{1}{z-1} - \frac{1}{1-2i}\,\frac{1}{z-2i} \qquad (*)$$

- In zone 2:

$$\frac{1}{z-1} = \frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = \sum_{k=1}^{\infty} z^{-k}$$

$$\frac{1}{z-2i} = \frac{-1}{2i}\,\frac{1}{1-\frac{z}{2i}} = \frac{-1}{2i}\sum_{k=0}^{\infty} \frac{z^k}{(2i)^k}$$

Substitute into $(*)$:

$$f(z) = \frac{1}{1-2i}\sum_{k=1}^{\infty} z^{-k} + \frac{1}{4+2i}\sum_{k=0}^{\infty} \frac{z^k}{(2i)^k}$$

How many ways have we learned to compute integral over a loop?

① Path parametrization:

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt$$

- Advantage: no need to worry about the function being holomorphic, as long as it is continuous on the curve.
- Drawback: integrand can easily become too complicated in terms of t.

② Fundamental thm. of Calc.

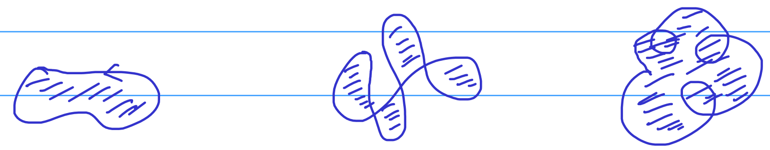
$$\int_\gamma f(z)\,dz = F(z)\Big|_{\gamma(\alpha)}^{\gamma(\beta)} = 0$$

- Advantage: no need to worry about parametrization of the loop.
- Drawback: sometimes antidervative DNE, or DNE in the region where the loop lies. Recall that for a function to have an antider. on a region, the function itself must be holomorphic there.

③ Cauchy-Goursat theorem:

$$\int_\gamma f(z)\,dz = 0$$

provided that f is holomorphic on the region enclosed by γ.

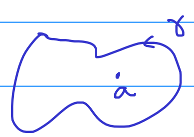


- Advantage: only need to check if the integrand is holo., not much to be concerned about the parametrization of the loop.

- Drawback: sometimes the loop encloses a singularity of the integrand.

④ Cauchy's Integral formula:

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$



γ: simple, possitively oriented

a inside γ

f holomorphic inside γ

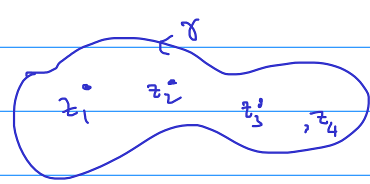
- Advantage: allows the integrand to have one singularity inside γ. Simple calculation.
- Drawback: doesn't allow integrand to be general fraction $\frac{f(z)}{g(z)}$

⑤ Cauchy's Residue theorem:

$$\int_\gamma f(z)\,dz = 2\pi i \left(\mathrm{Res}[f; z_1] + \cdots + \mathrm{Res}[f; z_n] \right)$$

γ: Simple loop, positively oriented

$z_1, z_2, \ldots, z_n$: isolated singularities of f enclosed in γ.



- Advantage: allows f to be quite general. Relatively simple calculation
- Drawback: doesn't work if f has non-isolated singularities inside γ.

Question: How to find the residues?

$\mathrm{Res}[f; z_0] = a_{-1}$ in the Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$

- If z_0 is removable, $a_{-1} = 0$.
- If z_0 is essential, compute a_{-1} by trying to write Laurent series.
- If z_0 is a pole of order n: (let's take $z_0 = 0$ for simplicity)

$$f(z) = a_{-n} z^{-n} + \cdots + a_{-1} z^{-1} + a_0 + a_1 z + \cdots$$

How do we know that z_0 is a pole of order n?

$\lim_{z \to 0} z^n f(z) = a_{-n}$, which is neither 0 nor ∞.

Ex

$$f(z) = \frac{e^z - 1}{z^2 \sin z}$$

Is $z=0$ a pole? What is its order?

We see that

$$e^z - 1 = z + \frac{z^2}{2} + \cdots = z\left(1 + \frac{z}{2} + \cdots\right)$$

$$z^2 \sin z = z^2\left(z - \frac{z^3}{6} + \cdots\right) = z^3\left(1 - \frac{z^2}{6} + \cdots\right)$$

Then

$$f(z) = \frac{z}{z^3} \frac{1 + z/2 + \cdots}{1 - z^2/6 + \cdots}$$

Thus, $z^2 f(z) \to 1$ as $z \to 0$.

$z = 0$ is a pole of order 2.

$$\rightsquigarrow \quad z^n f(z) = a_{-n} + a_{-n+1} z + \cdots + a_{-1} z^{n-1} + \cdots$$

We see that a_{-1} is the $(n-1)$'st coefficient of the Taylor series of function $g(z) = z^n f(z)$.

$$a_{-1} = \frac{1}{(n-1)!} g^{(n-1)}(0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\Big|_{z=0} (z^n f(z))$$

Another way to write it is:

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n f(z)\right)$$

Examples provided on worksheet.