Lecture 3 (4/5/2019)

* Geometric representation of complex numbers:

Because each complex number is associated with a pair of real numbers, it can be viewed as a vector in $\mathbb{R}^2$.

$$z = \underbrace{a+bi}_{\text{number}} \equiv \underbrace{(a,b)}_{\text{vector}} \quad \text{(duo nature of complex number)}$$

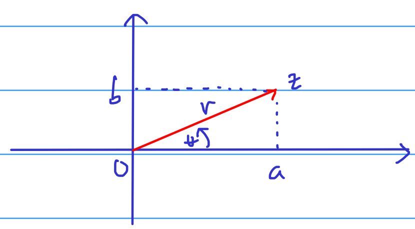
number $\updownarrow$ alg. operations; vector $\updownarrow$ distance } two notions needed for calc.

Caution:

$$\left.\begin{aligned} z &\equiv (a,b) \\ w &\equiv (c,d) \end{aligned}\right\} \quad zw \neq (a,b)\cdot(c,d)$$

$zw = (ac-bd,\ ad+bc)$, $(a,b)\cdot(c,d) = (ac+bd,\ 0)$

The complex multiplication is not the same as dot product.



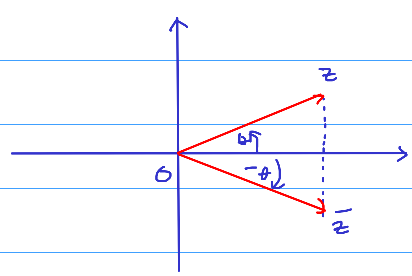
Argand diagram

$z = a+bi \equiv (a,b)$

$|z| = \sqrt{a^2+b^2} = r$

$\underbrace{\arg z}_{\text{multivalued function}} = \theta + k2\pi \quad (k \in \mathbb{Z})$

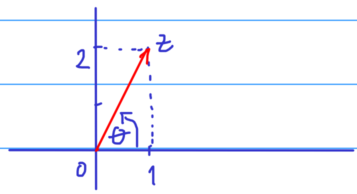
$\underbrace{\operatorname{Arg} z}_{\text{principal argument}} = \theta \in (-\pi, \pi]$



$\arg(-z) = \theta + \pi + k2\pi$

$\arg(\bar{z}) = -\theta + k2\pi$

Ex : $z = 1+2i$



$|z| = \sqrt{1^2+2^2} = \sqrt{5}$

$\tan\theta = \dfrac{2}{1} = 2$

$\theta = \arctan(2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$Re(z) = 1$

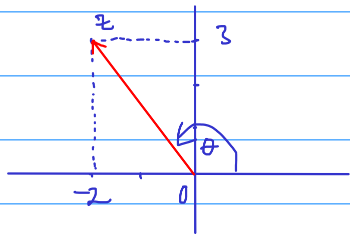
$Im(z) = 2$

$\bar{z} = 1-2i$

$Arg\, z = \arctan(2)$

$arg\, z = \arctan(2) + k2\pi$

Ex : $z = -2+3i$



$Re(z) = -2$

$Im(z) = 3$

$|z| = \sqrt{(-2)^2+3^2} = \sqrt{13}$

$\tan\theta = \dfrac{Im(z)}{Re(z)} = -\dfrac{3}{2}$

$\bar{z} = -2-3i$

$\theta = \underbrace{\arctan\left(-\frac{3}{2}\right)}_{\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)} + \pi$

$Arg\, z = \arctan(-3/2) + \pi$

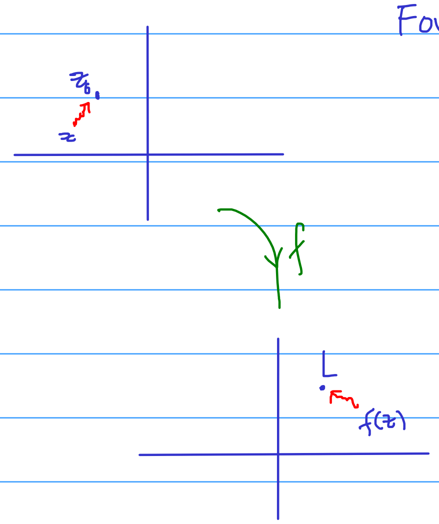
$arg\, z = \arctan(-3/2) + \pi + k2\pi$

* The notion of distance :

The distance between two complex numbers z and w is $|z-w|$

Let $f : \mathbb{C} \to \mathbb{C}$ (a map that takes a complex number to a complex number)

The notation $\lim_{z\to z_0} f(z) = L$ means



For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|z - z_0| < \delta$ then $|f(z) - L| < \varepsilon$.

(The notation of limit and continuity are the same as in vector calculus.)

We will see later that the notion of derivative is quite different compare to vector calculus (one can't divide by a vector, but can divide by a number).

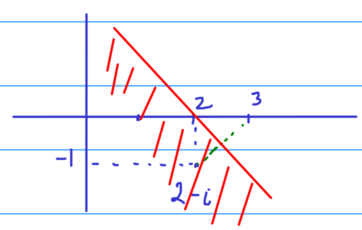
$$f'(z_0) = \underbrace{\lim_{z \to z_0}}_{\text{distance}} \underbrace{\frac{f(z) - f(z_0)}{z - z_0}}_{\text{alg.}}$$

the notion of derivative for complex-variable functions is more restrictive than real-variable functions.

Ex: Sketch the set $\{z : \underbrace{|z - 2 + i| < |z - 3|}\}$

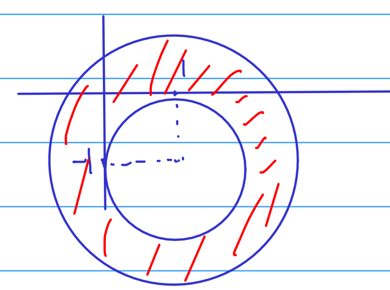
distance from z to $2-i$ is less than distance from z to 3.

$\leadsto$ z lies on one side of the perpendicular bisector of the line segment from $2-i$ to 3.



Ex:

the set $\{z : 1 < |z - 1 + i| < 2\}$



* Polar form of complex numbers:

$$\left.\begin{array}{l} r = |z| \\ \theta = \arg z \end{array}\right\} \text{ write } z = r\angle\theta = r(\cos\theta + i\sin\theta)$$

$$= r\,\mathrm{cis}(\theta)$$

$$= re^{i\theta}$$ ⇝ notation by Eulers reflecting surprising prop. of exponential function.

We will see that the exponential function can be used to construct trigonometric functions (a surprising fact from the standpoint of calc. of real variables).