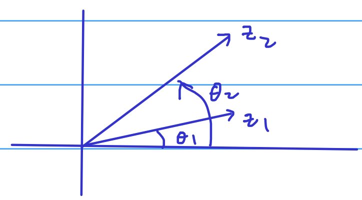
Lecture 4 (4/8/2019)

More examples on plotting with Mathematica (posted on course webpage).

Geometric nature of complex intertwines with alg. nature of complex numbers.

$$z_1 = r_1 \angle \theta_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2 \angle \theta_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$



$$z_1 z_2 = r_1 r_2 \Big[\underbrace{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)}_{\cos(\theta_1+\theta_2)} + i\,\underbrace{(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)}_{\sin(\theta_1+\theta_2)} \Big]$$

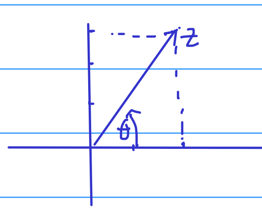
- Rule of thumb: to multiply two complex numbers, we mult. their moduli and add their arguments.

Note: The argument behaves like logarithm function: arg. of a product = sum of the arguments.

Ex: $z = 2+3i$

How to find a formula for z^n?

First, convert z into polar form:



$$r = \sqrt{2^2+3^2} = \sqrt{13}$$

$$\tan\theta = \frac{3}{2}$$

$$\theta = \arctan\left(\frac{3}{2}\right) = \text{an explicit number}$$

$$z = r\,\mathrm{cis}(\theta)$$

$$z^n = r^n\,\mathrm{cis}(n\theta) = 13^{\frac{n}{2}}(\cos n\theta + i\sin n\theta)$$

$$(2+3i)^n = 13^{\frac{n}{2}} \cos n\theta + i\, 13^{\frac{n}{2}} \sin n\theta$$

Special case: $\text{cis}^n(\theta) = \text{cis}(n\theta)$ (called de Moivre's formula)

Application: express $\cos 2\theta$, $\sin 2\theta$ in terms of $\cos\theta$ and $\sin\theta$.

$$\underbrace{(\cos\theta + i\sin\theta)^2}_{\text{Expand:}} = \cos 2\theta + i\sin\theta$$

$$\cos^2\theta - \sin^2\theta + i\,2\cos\theta\sin\theta$$

Equate real parts: $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Equate imaginary parts: $\sin 2\theta = 2\cos\theta\sin\theta$.

* n'th roots:

$$z = 2+3i$$

$$\sqrt[8]{z} = ?$$

- Solve for x and y such that $(x+iy)^8 = 2+3i$
- Solve for ρ and α such that $(\rho\,\text{cis}\,\alpha)^n = \rho^8\,\text{cis}(8\alpha) = 2+3i$

Write $2+3i$ in polar form: $2+3i = r\,\text{cis}\,\theta$

where $r = \sqrt{2^2+3^2} = \sqrt{13}$

$$\theta = \arctan\left(\frac{3}{2}\right)$$

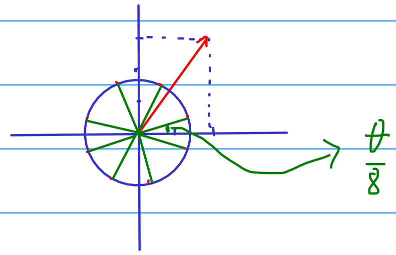
Want $\rho^8\,\text{cis}(8\alpha) = r\,\text{cis}\,\theta$

Thus,

$$\begin{cases} \rho^8 = r = \sqrt{13} \\ 8\alpha = \theta + k2\pi \end{cases} \rightsquigarrow \begin{cases} \rho = \sqrt[8]{r} = \sqrt[16]{13} \\ \alpha = \frac{\theta}{8} + k\frac{2\pi}{8} \end{cases}$$

There are 8 different values for α, corresponding to $k=0,1,\dots,7$

$$\alpha \in \left\{ \frac{\theta}{8},\ \frac{\theta}{8}+\frac{2\pi}{8},\ \frac{\theta}{8}+\frac{6\pi}{8},\ \dots,\ \frac{\theta}{8}+\frac{14\pi}{8} \right\}$$



<u>Thm</u>: a number $z \neq 0$ has exactly n distinct n'th roots.

<u>Ex</u>: $\sqrt[4]{-1}$

$$\left.\begin{aligned} |-1| &= 1 \\ \operatorname{Arg}(-1) &= \pi \end{aligned}\right\} -1 = 1\,\mathrm{cis}(\pi)$$

$$\sqrt[4]{-1} = \left\{ \sqrt[4]{1}\ \mathrm{cis}\frac{\pi + k2\pi}{4},\quad k=0,1,2,3 \right\}$$

$$= \left\{ \mathrm{cis}\left(\frac{\pi}{4} + k\frac{\pi}{2}\right),\quad k=0,1,2,3 \right\}$$

$$= \left\{ \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right),\ \cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right),\right.$$

$$\left. \cos\left(\frac{\pi}{4}+\pi\right) + i\sin\left(\frac{\pi}{4}+\pi\right),\ \cos\left(\frac{\pi}{4}+\frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\pi}{2}\right) \right\}$$

$$= \left\{ \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\ \frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\ \frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right\}$$