Lecture 5 (4/10/2019)

Ex: Solve for complex roots of $2z^2 - 3z + 2 = 0$

Recall: $ax^2 + bx + c = 0$ has roots $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(Why? Completing the square: $0 = x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}$)

Now the case $b^2 - 4ac < 0$ is no longer an obstacle.

Back to the example: $z = \dfrac{3 \pm \sqrt{-7}}{4}$

There are two square roots of -7: $\sqrt{-7} = \{\sqrt{7}\,i, -\sqrt{7}\,i\}$

$$z = \frac{3 \pm \sqrt{7}\,i}{2}$$

Ex: $z^6 + z^3 + 1 = 0$ $\leftarrow$ expect 6 roots (counted with multiplicities)

Put $w = z^3$, then $w^2 + w + 1 = 0$

Use quadratic formula to compute w:

$$w = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$w_1 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \operatorname{cis}\left(\frac{4\pi}{3}\right)$$

$$w_2 = \overline{w_1} = \operatorname{cis}\left(-\frac{4\pi}{3}\right) = \operatorname{cis}\left(\frac{2\pi}{3}\right)$$

* Solve for z such that $z^3 = w_1$:

$$z \in \left\{ \operatorname{cis}\left(\frac{4\pi/3 + k2\pi}{3}\right) : k = 0, 1, 2 \right\}$$

$$= \left\{ \underbrace{\operatorname{cis}\left(\frac{4\pi}{9}\right)}_{z_1}, \underbrace{\operatorname{cis}\left(\frac{10\pi}{9}\right)}_{z_2}, \underbrace{\operatorname{cis}\left(\frac{16\pi}{9}\right)}_{z_3} \right\}$$

* Solve for z such that $z^3 = w_2$:

$$z \in \left\{ \operatorname{cis}\left(\frac{2\pi/3 + k2\pi}{3} \right) : k = 0, 1, 2 \right\}$$

$$= \left\{ \underbrace{\operatorname{cis}\left(\frac{2\pi}{9}\right)}_{z_4}, \underbrace{\operatorname{cis}\left(\frac{8\pi}{9}\right)}_{z_5}, \underbrace{\operatorname{cis}\left(\frac{14\pi}{9}\right)}_{z_6} \right\}$$

How to define functions e^z, $\cos z$, $\sin z$, $\log z$, ... for complex variables?

Recall how one defined function e^x with real variable x.

Start with $a, a^2, a^3, a^4, \ldots$

Then $a^{-1} := \frac{1}{a}$, $a^{-2} := \frac{1}{a^2}$, ...

Then $a^{1/2} = \sqrt{a}$, $a^{1/3} = \sqrt[3]{a}$, ...

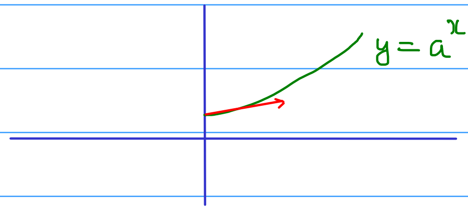
Note that this notation ($\sqrt{}$) doesn't suggest how to "compute" the number. For example, $\sqrt{2}$ denotes a number $x > 0$ such that $x^2 = 2$. This number exists due to the continuity of function $f(x) = x^2$ (using Intermediate value theorem). But one needs further tools (limit, Taylor polynomials) than just algebra to compute $\sqrt{2}$.

Then $a^{-2/3}$, $a^{-3/2}$, $a^{-5/3}$, ...

$\rightsquigarrow$ one has defined a^r for any $r \in \mathbb{Q}$.

For each $x \in \mathbb{R}$, take a sequence $r_n \in \mathbb{Q}$ converging to x.

Define $a^x := \lim_{n \to \infty} a^{r_n}$.



e is the value of a such that the slope of the curve $y = a^x$ at $x = 0$ is equal to 1.

To define 2^z for $z \in \mathbb{C}$, the same procedure works if one knows how to define a^i. More specifically,

$$2^{x+iy} := 2^x (2^y)^i$$

How to define e^i?

There is another way to define e^x (from computational perspective):

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

(Ingredient needed: algebraic operations and the notion of limit
$\rightsquigarrow$ these are available on $\mathbb{C}$ as well.)

Formally plug $x = i$:

$$\text{"}e^i\text{"} = 1 + \frac{i}{1} + \frac{i^2}{2} + \frac{i^3}{6} + \frac{i^4}{24} + \frac{i^5}{120} + \cdots$$

$$= \left(1 - \frac{1}{2} + \frac{1}{24} - \cdots\right) + i\left(\frac{1}{1} - \frac{1}{6} + \frac{1}{120} - \cdots\right)$$

Let's look at a more general picture: replace x by ix:

$$\text{"}e^{ix}\text{"} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \cdots$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}_{\text{power series of cosine}} + i\underbrace{\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}_{\text{power series of sine}}$$

$$= \cos x + i\sin x$$

$$= \operatorname{cis} x$$

Define $e^{ix} := \operatorname{cis} x \qquad (x \in \mathbb{R})$

Plug $x = \pi$:

$$e^{i\pi} = \operatorname{cis} \pi = \cos\pi + i\sin\pi = -1$$

$$\rightsquigarrow \quad e^{i\pi} + 1 = 0$$

This is called Euler's formula, which shows how the five important numbers $0, 1, \pi, e, i$ are related.