Lecture 6 (4/12/2019)

Recall that we defined $e^{ix} = \cos x + i\sin x = \text{cis}\, x$ for all $x \in \mathbb{R}$.

Formally, $e^{x+iy} = e^x e^{iy} = e^x \text{cis}(y)$.

Definition:

$$\boxed{e^z = \underbrace{e^x \text{cis}(y)}_{\text{polar form}} = \underbrace{e^x \cos y + i e^x \sin y}_{\text{standard}}.}$$

Ex:

$$e^{2+3i} = \underbrace{e^2 \text{cis}(3)}_{\text{polar form}} = \underbrace{e^2 \cos 3 + i e^2 \sin 3}_{\text{standard form}}$$

$$e^{\pi i} = e^0(\cos\pi + i\sin\pi) = -1$$

$$e^{z+2\pi i} = e^{x+(y+2\pi)i} = e^x \text{cis}(y+2\pi) = e^x \text{cis}(y) = e^z$$

Thus, e^z is periodic function with period $2\pi i$.

$\rightsquigarrow$ Complex exponential is not one-to-one as for real exp.

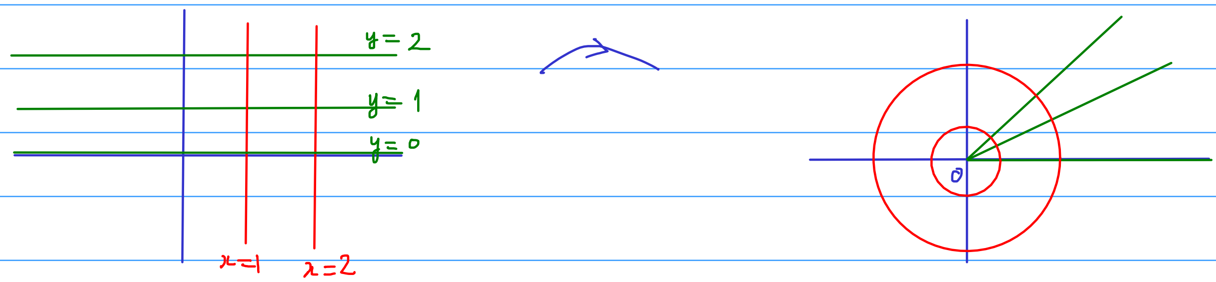
Check the identity $e^{z+w} = e^z e^w$?

$$\left.\begin{aligned} z &= a + ib \\ w &= c + id \end{aligned}\right\} \quad z+w = (a+c) + i(b+d)$$

$$\left.\begin{aligned} e^z &= e^a \text{cis}\, b \\ e^w &= e^c \text{cis}\, d \end{aligned}\right\} \quad e^z e^w = e^a e^c (\text{cis}\, b)(\text{cis}\, d) = e^{a+c} \text{cis}(b+d)$$

$$= e^{z+w}$$

Let's look at some geometric properties of the map $z \mapsto e^z$:



The line $y = 1$ has parametric eq. $(x, y) = (t, 1)$

Image under e^z: $\quad (e^x \cos y, e^x \sin y) = (\underbrace{e^t \cos 1, e^t \sin 1}_{})$

this is a half line through the origin with slope $\tan 1$.

The map e^z maps every line parallel to the horizontal axis the a half line (ray) through the origin.

The line $x = 1$ has parametric equation $(x, y) = (1, t)$.

Image under e^z:

$e^1 \operatorname{cis}(t) = e \operatorname{cis}(t)$ ----- a circle centered at the origin with radius e.

e^z maps every line parallel to the vertical axis to a circle centered at the origin.

How to define $\cos z$ and $\sin z$?

Formally, $\cos(x+iy) \text{ "=" } \cos x \cos iy - \sin x \sin iy$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos ix := 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

Similarly,
$$\sin ix := \frac{e^{-x} - e^{x}}{2i} = i \sinh x$$

With these definitions,

$$\begin{aligned}\cos z = \cos(x+iy) &:= \cos x \cos iy - \sin x \sin iy \\ &= \cos x \frac{e^{y}+e^{-y}}{2} + i \sin x \frac{e^{y}-e^{-y}}{2} \\ &= \frac{e^{iz} + e^{-iz}}{2} \quad \text{(easy to check)}\end{aligned}$$

<u>Definition</u>:

$$\boxed{\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}}$$

<u>Ex</u>:

$$\begin{aligned}\cos(1+2i) &= \frac{1}{2}\left(e^{i(1+2i)} + e^{-i(1+2i)}\right) = \frac{1}{2}\left(e^{-2+i} + e^{2-i}\right) \\ &= \frac{1}{2}\left(e^{-2} \operatorname{cis} 1 + e^{2} \operatorname{cis}(-1)\right) \\ &= \frac{1}{2}\left(e^{-2}\cos 1 + e^{2} \cos 1\right) + i\frac{1}{2}\left(e^{-2}\sin 1 - e^{2} \sin 1\right)\end{aligned}$$