Lecture 7 (4/15/2019)

$$e^z = e^{\operatorname{Re} z}\operatorname{cis}(\operatorname{Im} z)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

} similar construction as e^z, $\cos z$, $\sin z$ (start with Taylor series for real variable $\cosh x$, $\sinh x$. Then formally plug ix for x)

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\,\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}, \qquad \cot z = \frac{1}{\tan z}$$

How to define logarithm?

$$\log z = w \quad \text{if} \quad e^w = z$$

Since exponential is not one-to-one, logarithm is a multi-valued function.

Solve for w: write $w = a + ib$

$$z = re^{i\theta}$$

$$e^w = e^a e^{ib} = re^{i\theta}$$

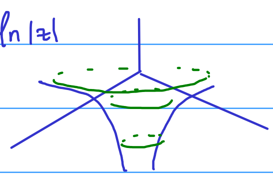
$$\rightsquigarrow \begin{cases} e^a = r \\ b = \theta + k2\pi \end{cases} \rightsquigarrow \begin{cases} a = \ln r \text{ (real logarithm)} \\ b = \theta + k2\pi \end{cases}$$

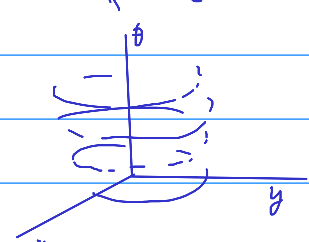
The equation $e^w = z$ has no solution $w \in \mathbb{C}$ only if $z = 0$. Therefore, the range of e^z is $\mathbb{C} \setminus \{0\}$.

Thus, $\log z = \ln r + i(\theta + k2\pi) = \ln|z| + i \arg z$

Visualize the real and imaginary part of logarithm:

$z \mapsto \ln|z|$

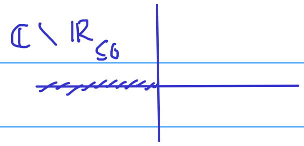


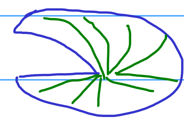


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ \theta = \theta \end{cases}$$

How to make logarithm a function?

$\rightsquigarrow$ cut a branch of argument, for example the principal branch, where the negative real line $\mathbb{R}_{\leq 0}$ is removed.





$\leftarrow$ the graph of function $z \longmapsto \operatorname{Arg}(z)$

$$\boxed{\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z} \leftarrow \text{the principal branch of logarithm}$$

Ex:

$\log i = ?$ $\quad$ $\operatorname{Log} i = ?$

$$\left.\begin{array}{l} |i| = 1 \\ \arg i = \frac{\pi}{2} + k2\pi \end{array}\right\} \rightsquigarrow \log i = \underbrace{\ln 1}_{\text{real}} + i\left(\frac{\pi}{2} + k2\pi\right) = i\left(\frac{\pi}{2} + k2\pi\right)$$

$$\operatorname{Log} i = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}.$$

Ex:

$$\log e^i = \ln |e^i| + i \arg(e^i) = \ln 1 + i(1 + k2\pi) = i(1 + k2\pi)$$

$(e^i = e^0 \operatorname{cis} 1)$

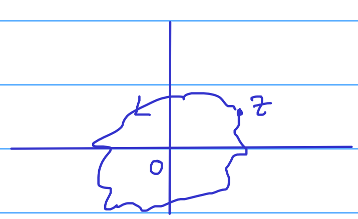
Another way: i is a value of $\log e^i$. Thus, $\log e^i = \{ i + k2\pi i : k \in \mathbb{Z} \}$.

* Comments on branches of the logarithm:

The idea to introduce branches is to make sure that a multi-valued function is single-valued and "continuous" (i.e. without jump).

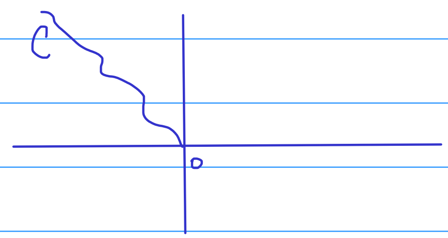
Logarithm is multi-valued because the argument is multivalued.

Without graphing the surface of $\arg z$, one can still tell that $\arg z$ is not "continuous" on $\mathbb{C} \setminus \{0\}$, unless one "cuts" the complex plane:



Consider a loop passing z and enclosing 0 as in the picture. The argument increases in value as one moves from z along the curve. When one arrives at z again, the argument increases by 2π. This is a jump (or "discontinuity").

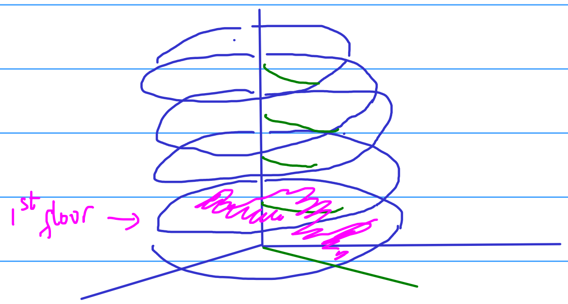
⇝ To avoid such jump (or, to make argument a well-defined single-valued function), one needs to make sure that all curves enclosing 0 are excluded.



One way to do so is to cut the plane $\mathbb{C}$ by a curve C starting at 0 going to infinity and not intersecting itself.

(0 in this case is called a branch point.
C " " " is called a branch cut.)

A branch cut cuts the surface of $\arg z$ into infinitely many branches (think of floors in a building).

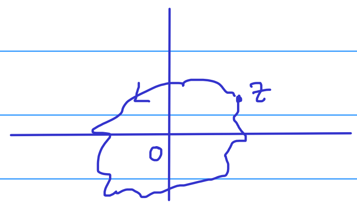


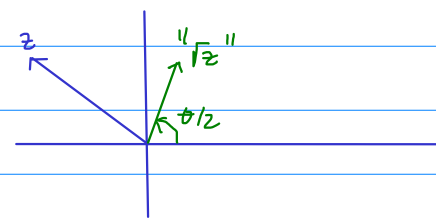
Power functions

$f(z) = \sqrt{z}$ ---- multi-valued function

$z = re^{i\theta}$

Formally, $\sqrt{z} = \sqrt{r}\, e^{i\frac{\theta}{2}}$





As z moves along a curve (on the left picture), the argument increases. As z goes back to the original position, the arg. increases by 2π. Thus, $\theta/2$ increases by π, which flips the sign of $\sqrt{r}\, e^{i\theta/2}$.

A branch cut is needed. Choose for example $C = \mathbb{R}_{\leq 0}$.
(the negative real line).

Definition:

$$z^a := e^{a \operatorname{Log} z}$$

where $\operatorname{Log} z$ is the principal logarithm.

Implicitly, the definition uses the branch cut $C = \mathbb{R}_{\leq 0}$.

Ex:

$$i^{1/2} = e^{\frac{1}{2}\operatorname{Log} i} = \exp\left(\frac{1}{2} i \frac{\pi}{2}\right) = \exp\left(i\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$(-1)^{1/3} = e^{\frac{1}{3}\operatorname{Log}(-1)} = \exp\left(\frac{1}{3} i\pi\right) = \exp\left(i\frac{\pi}{3}\right) = \frac{1}{3} + i\frac{\sqrt{3}}{2}$$