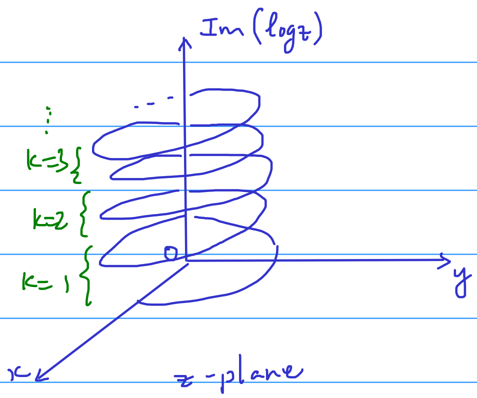
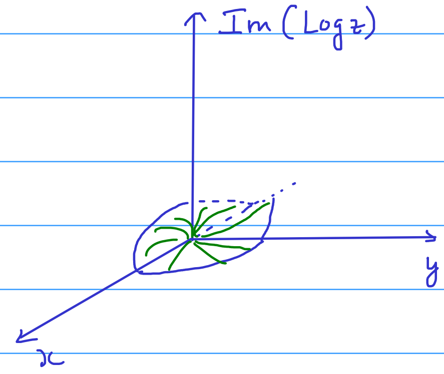
Lecture 8 (4/17/2019)

* Recall: $\log z = \ln|z| + i \arg z$

$= \ln|z| + i(\mathrm{Arg}\, z + k2\pi)$ (multi-valued function)

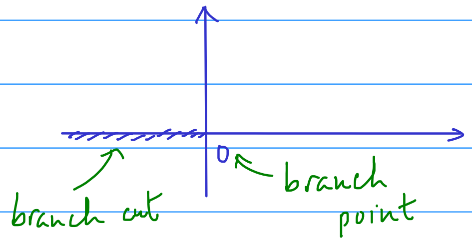
$\mathrm{Log}\, z = \ln|z| + i \underbrace{\mathrm{Arg}\, z}_{\in (-\pi, \pi]}$ (principal branch of the logarithm)





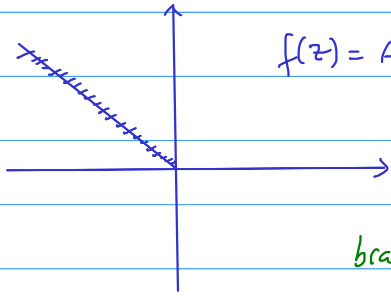
If $z \in \mathbb{R}$ and $z > 0$, $\mathrm{Log}\, z = \ln z$.
The logarithm can be used to define various function such as z^a, $\arcsin z$, $\arccos z$, ... These are multi-valued functions.
Usually a branch cut is used to create a region on z-plane such that the function is single-valued (so that one can do calculus on it).

Ex: $\mathrm{Arg}\, z$, $\mathrm{Log}\, z$ are well-defined functions on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

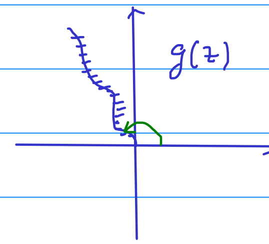


There are infinitely many possible choices of branch cut for $\arg z$.
However, each must start at 0 and go to infinity (to rule out

any possible curve enclosing 0).



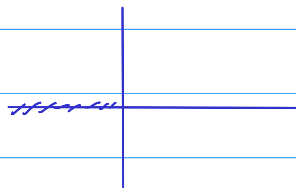
$f(z) = \operatorname{Arg} z + \frac{\pi}{4} \in \left(-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$



branch cuts of arg z

Ex: What is the domain of $f(z) = \operatorname{Log} z$?

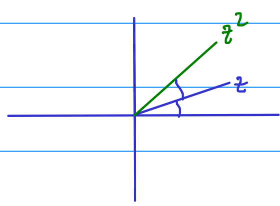
$\mathbb{C} \setminus \mathbb{R}_{\leq 0} = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0 \text{ and } z \leq 0\}$



Ex: What is the domain of $g(z) = \operatorname{Log}(z^2)$?

It consists of all complex numbers z except for those that satisfy $z^2 \leq 0$.

$\operatorname{Arg} z = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ implies that $\operatorname{Arg} z^2 = \pi$.



Thus, the domain of z is everything but the imaginary axis: $\mathbb{C} \setminus i\mathbb{R} = \mathbb{C} \setminus \{z : z = it, t \in \mathbb{R}\}$.

* Note: $\operatorname{Log} z^2 \neq 2 \operatorname{Log} z$ since the two functions have different domains.

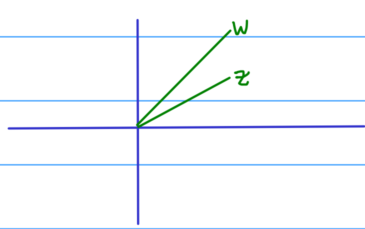
More generally,

$$\operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w \quad \text{in general}$$

This is because

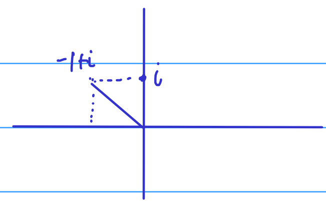
$$\operatorname{Arg}(zw) \neq \operatorname{Arg} z + \operatorname{Arg} w.$$

(They are equal only in modulo 2π.)



If z and w lie on the right half plane (that is, $\operatorname{Re} z$ and $\operatorname{Re} w$ are positive), then

$$\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w.$$

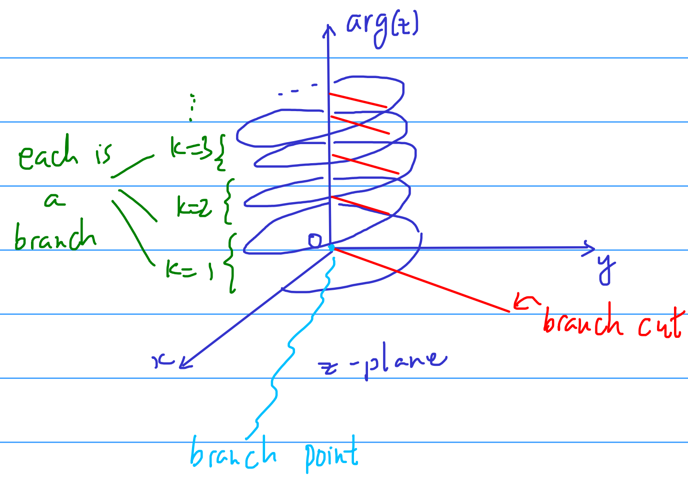


$$\arg(i(-1+i)) = \frac{\pi}{2} + \frac{3\pi}{4} + k2\pi = \frac{5\pi}{4} \pmod{2\pi}$$

$$\rightsquigarrow \operatorname{Arg}(i(-1+i)) = -\frac{3\pi}{4} \in (-\pi, \pi]$$

$$\operatorname{Arg} i = \frac{\pi}{2}, \quad \operatorname{Arg}(-1+i) = \frac{3\pi}{4}$$

Although each point $z \in \mathbb{C}\setminus\{0\}$ has an argument (or more precisely, infinitely many arguments, each differing from another by a multiple of 2π), it is impossible to define a single-valued continuous argument function on $\mathbb{C}\setminus\{0\}$.



There are two strategies to define a function from a multi-valued function.

1) If the function involves logarithm or argument, use principal branch of logarithm (Log) or of argument (Arg).
2) Identify branch points, introduce branch cuts, and pick a branch.

Ex: $\log(z^2-1)$

The cause of multi-value is the logarithm. A single-valued function can be defined by taking the principal branch of logarithm: $f(z) = \operatorname{Log}(z^2-1)$.
Note that the domain of f has to be carefully calculated:

$$\mathbb{C}\setminus\{z\in\mathbb{C} : z^2-1 \le 0\}$$

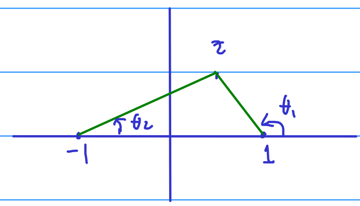
Ex: $\sqrt{z^2-1}$, understood as multi-valued function $e^{\frac{1}{2}\log(z^2-1)}$.

One can define a single-valued function $\sqrt{z^2-1}$ by taking the principal branch of logarithm. Here is another method:

First, write $\sqrt{z^2-1} = \sqrt{(z-1)(z+1)}$

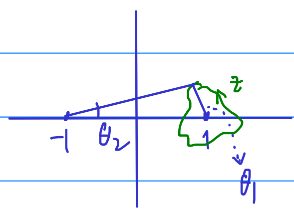
$\theta_1 = \operatorname{Arg}(z-1) \in (-\pi, \pi]$

$\theta_2 = \operatorname{Arg}(z+1) \in (-\pi, \pi]$



One writes formally that

$$\operatorname{Arg} \text{"}\underbrace{\sqrt{z^2-1}}_{\text{not defined yet}}\text{"} = \frac{\theta_1+\theta_2}{2} \in (-\pi, \pi]$$

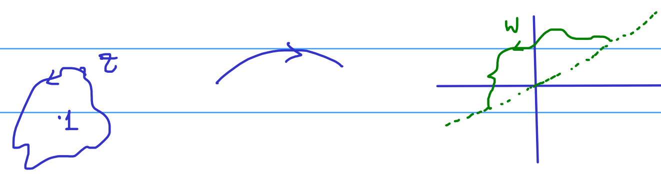


As z moves along a curve enclosing 1,

$$\begin{cases} \theta_1 \text{ increases by } 2\pi \\ \theta_2 \text{ remains the same} \end{cases}$$

at the time z comes back to the original position.

$\frac{\theta_1+\theta_2}{2}$ increases by π, which implies a jump in value.

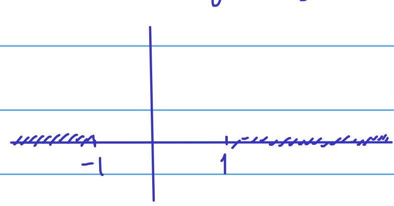


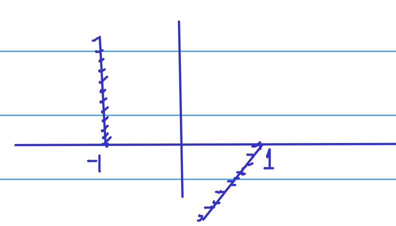
Thus, 1 is a branch point. Similarly, -1 is also a branch point.

There are no other branch points.

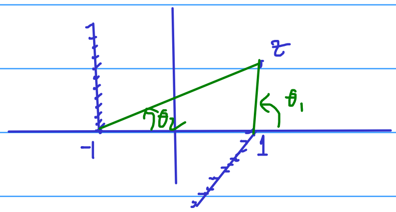
Next, introduce branch cuts to forbid curves from enclosing -1 and 1.

There are many ways to do so. For example,





Next, define a branch (that is, to specify the range of θ_1 and θ_2).



$$\left.\begin{array}{l} \theta_1 \in \left(-\frac{3\pi}{4}, \frac{5\pi}{4}\right] \\ \theta_2 \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right] \end{array}\right\} \text{one branch}$$

$$\left.\begin{array}{l} \theta_1 \in \left(\frac{5\pi}{4}, \frac{13\pi}{4}\right] \\ \theta_2 \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right] \end{array}\right\} \text{another branch}$$

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