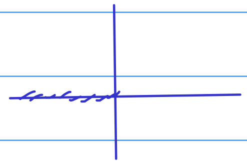
Lecture 9 (4/19/2019)

Ex: find domain of $f(z) = \sqrt{1-z^4}$ (principal branch being used).



We need to exclude any point $z \in \mathbb{C}$ such that

$$1 - z^4 \le 0.$$

That is, $z^4 \geq 1$

In particular, $\operatorname{Arg}(z^4) = 0$

We know that $\operatorname{Arg}(z^4) = 4 \operatorname{Arg} z \pmod{2\pi}$

$\rightsquigarrow$ $4 \operatorname{Arg} z = 0 \pmod{2\pi}$

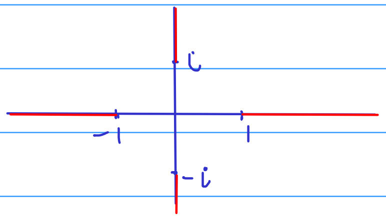
$\rightsquigarrow$ $4 \operatorname{Arg} z = k 2\pi$

$\rightsquigarrow$ $\operatorname{Arg} z = k \frac{\pi}{2}$

$\rightsquigarrow$ $\operatorname{Arg} z \in \{0, \frac{\pi}{2}, \pi, -\frac{\pi}{2}\}$

$\rightsquigarrow$ z lies on the real and imaginary axes.

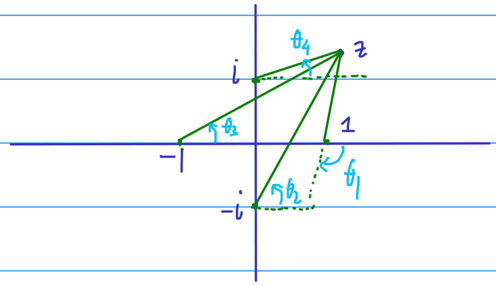
$z^4 \geq 1$ $\rightsquigarrow$ the modulus of z is ≥ 1.



Domain $= \mathbb{C} \setminus \{ z : z = a$ with $a \in \mathbb{R}, |a| \geq 1,$
or $z = ia$ with $a \in \mathbb{R}, |a| \geq 1 \}$.

One sees that the 4 points $\pm 1, \pm i$ are branch points of the multivalued function $\sqrt{1-z^4}$. Indeed, write

$$\sqrt{1-z^4} = \sqrt{(1-z)(z+1)(z-i)(z+i)}$$



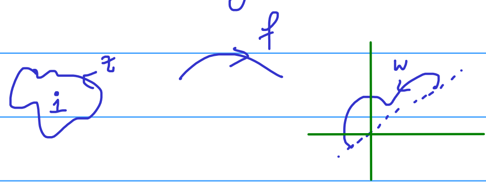
$\theta_1 = \operatorname{Arg}(1-z)$

$\theta_2 = \operatorname{Arg}(z+1)$

$\theta_3 = \operatorname{Arg}(z-i)$

$\theta_4 = \operatorname{Arg}(z+i)$

As z moves along a small closed curve enclosing 1,



θ_1 increases by 2π,

θ_2 returns to its original value,

θ_3 " "

θ_4 " "

The quantity $\dfrac{\theta_1+\theta_2+\theta_3+\theta_4}{2}$ increases by π, which means $w=f(z)$ does not return to it original position. The branch cuts (the red lines above) forbid any closed curve in the domain to enclose any of the branch points $\pm i, \pm 1$.

Ex: find the domain of $f(z) = \sqrt{1+z}\,\sqrt{i-z}$ (principal branch)

Remove all z's such that $1+z \le 0$ or $i-z \le 0$

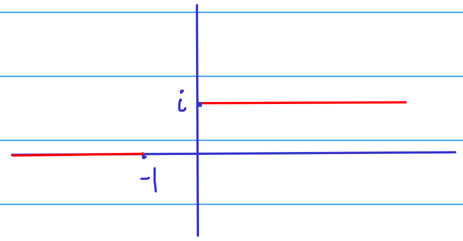
For the first part, $z \le -1$.

For the second part, $i - z = a \le 0$

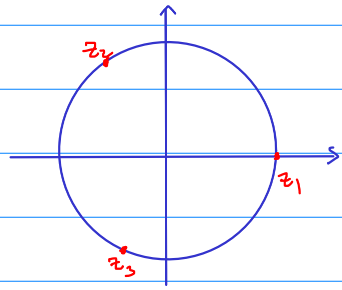
$\rightsquigarrow z = i - a \qquad (a \le 0)$

$\rightsquigarrow z = i + b \quad (b \ge 0)$

this is equation of a line $y = 1$



Ex: find a branch $f(z)$ of $z^{1/3}$ such that $f(1) = \angle \frac{2\pi}{3}$



If the principal branch of logarithm were chosen, $f(1)$ would be $z_1 = 1$.

$$f(z) = e^{\frac{1}{3}\log z} = e^{\frac{1}{3}(\ln|z| + i\,\mathrm{Arg}\,z + ik2\pi)}$$
$$= |z|^{1/3} e^{(i\,\mathrm{Arg}\,z + ik2\pi)/3}$$
$$(k = 0, 1, 2)$$

Plug $z = 1$: $f(1) = e^{(i0 + ik2\pi)/3} = e^{i\frac{k2\pi}{3}}$

It was given that $f(1) = e^{i\frac{2\pi}{3}}$. Thus, $k = 1$.

Here is the definition of f:

$$f(z) = |z|^{1/3} e^{i\,\mathrm{Arg}\,z + i2\pi/3} = \underbrace{z^{1/3}}_{\text{principal branch}} e^{i\frac{2\pi}{3}}$$

In other word, $f(z)$ differs from the principal branch of $z^{1/3}$ by a rotation of $2\pi/3$ counter-clockwise.

*Recall: $z^a := e^{a\,\mathrm{Log}\,z}$

Special case: $a \in \mathbb{R}$. In this case,

$$z^a = e^{a(\ln|z| + i\,\mathrm{Arg}\,z)} = e^{a\ln|z|} e^{ia\,\mathrm{Arg}\,z}$$
$$= |z|^a e^{ia\,\mathrm{Arg}\,z}$$

Rule of thumb: z^a $(a \in \mathbb{R})$ $\begin{cases} \text{raise modulus to power } a, \\ \text{multiply principal arg. by } a. \end{cases}$

(similar to positive integer power)

* Properties:

1) $$z^{a+b} = z^a z^b$$

using the same branch of logarithm

Why?

$$LHS = e^{(a+b)\,\mathrm{Log}\,z} = e^{a\,\mathrm{Log}\,z + b\,\mathrm{Log}\,z} = e^{a\,\mathrm{Log}\,z}\, e^{b\,\mathrm{Log}\,z} = z^a z^b$$

2) $$z^{-a} = \frac{1}{z^a}$$

3) $$z^{a-b} = \frac{z^a}{z^b}$$

4) $(z^a)^n = z^{an}$ if n is integer

This equality is not necessarily true if n is complex.

* Caution:

$(zw)^a \neq z^a w^a$ in general

It comes from the fact that $\mathrm{Log}(zw) \neq \mathrm{Log}\, z + \mathrm{Log}\, w$, which comes from the fact that

$$\mathrm{Arg}(zw) \neq \mathrm{Arg}\, z + \mathrm{Arg}\, w$$