Multivalued functions via Mathematica

Among many multivalued complex functions are the argument function $\arg z$, the logarithm $\log z$, the inverse sine $\arcsin z$, inverse cosine $\arccos z$, and the power function z^a (a is non-integer). These are not functions in usual sense. One can only do calculus on a portion, called *single-valued branch* or simply *branch*, of a multi-valued function.

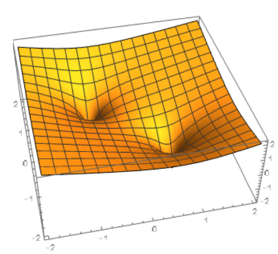
As you have seen in Homework 2, a single-valued function $f(z)$ can be visualized by sketching its real part $\operatorname{Re} f(z)$, imaginary part $\operatorname{Im} f(z)$, modulus $|f(z)|$, principal argument $\operatorname{Arg} f(z)$ separately. Likewise, a multivalued function can also be visualized by sketching its real part, imaginary part, modulus, argument separately. However, each of these parts itself can be multivalued. For example, function $\log z$ has multivalued imaginary part, function $z^{1/3}$ has multivalued argument, function $\arg z$ has multivalued real part (with zero imaginary part). From Homework 2, you already see that the "graph" of $\arg z$ looks like a parking deck with infinitely many floors.

This note will help you use Mathematica to visualize more general multivalued functions, their branches, branch cuts, and branch points. Let us consider a multivalued function $\log(z^2 + i)$. By the definition of logarithm,

$$\log(z^2 + i) = \ln|z^2 + i| + i \arg(z^2 + i).$$

The real part of this function is single-valued. One can graph it using command **Plot3D** in Mathematica:

```
Plot3D[Log[Abs[(x + y*I)^2 + I]], {x, -2, 2}, {y, -2, 2}]
```



The imaginary part is multivalued

$$f(z) = \arg(z^2 + i).$$

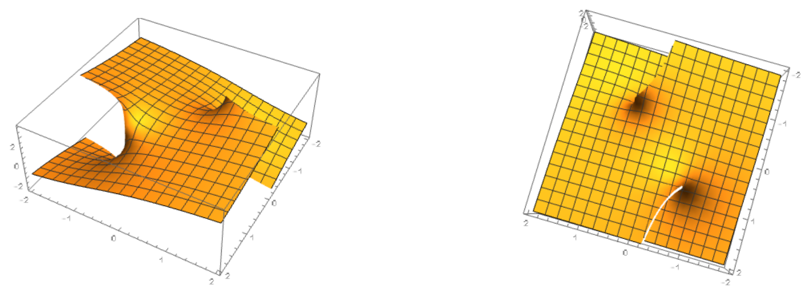
It is the composite of multivalued function $\arg w$ and single-valued function $z^2 + i$. Here we use the name w instead of z to avoid confusion. Function $f(z)$ would become single-valued if we specify a branch for $\arg w$. This is done by restricting the range of function $\arg w$ to, for example, $(-\pi, \pi]$. One obtains a branch for $\arg w$ (the principal branch, denoted by $\operatorname{Arg} w$), and thereby obtains a branch for function $f(z)$, namely

$$F(z) = \operatorname{Arg}(z^2 + i).$$

One can sketch $F(z)$ by the command **Plot3D** in Mathematica:

```
Plot3D[Arg[(x + y*I)^2 + I], {x, -2, 2}, {y, -2, 2}]
```

This is the graph of only one branch of $f(z)$. Other branches come from branches of $\arg w$ other than $\operatorname{Arg} w$, namely $\operatorname{Arg} w + k2\pi$ where $k \in \mathbb{Z}$. Therefore, all branches of $f(z)$ are $\operatorname{Arg}(z^2+i) + k2\pi$, which is $F(z) + k2\pi$. For each k, one draws the graph of $F(z) + k2\pi$. The combination of all these graphs gives a full picture of $f(z)$. In other words, the "graph" of the multivalued function $f(z)$ is a concatenation of copies of the graph of $F(z)$. Each copy is a vertical shift by a multiple of 2π of the graph of $F(z)$. In Mathematica,



(a) Looking from an angle (b) Looking from above

Figure 1: Graph of $F(z)$

```
p[k_] := Plot3D[Arg[(x + y*I)^2 + I] + k*2*Pi, {x, -2, 2}, {y, -2, 2}]
Show[p[0], p[1], PlotRange -> All]
```

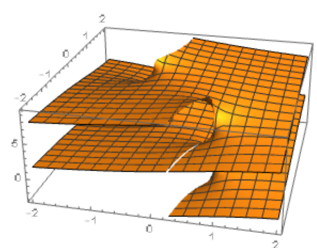


Figure 2: Two branches of multivalued function $f(z)$

Now look back to Figure 1. There are two curves on the complex plane where function $F(z)$ exhibits jumps (discontinuity). These jumps are due to discontinuity of function $\operatorname{Arg} w$ across the negative real axis $\mathbb{R}_{\leq 0}$. It is usually preferable to do calculus with continuous functions. Although $\operatorname{Arg} w$ is well-defined on $\mathbb{C}\backslash\{0\}$, it is only continuous on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$. By forbidding w from lying on $\mathbb{R}_{\leq 0}$, one obtains a continuous restriction of $\operatorname{Arg} w$. The ray $\{\arg w = \pi \pmod{2\pi}\}$ is a *branch cut* of $\arg w$. *A branch cut is used to create a continuous single-valued branch for a multivalued function.* Once this branch cut is applied, it induces a branch cut for $f(z)$. Indeed, all z's such that $z^2 + i \in \mathbb{R}_{\leq 0}$ have to be removed from the domain of $f(z)$. From Figure 1 (b), we see that these removed points form two curves (where discontinuity occurs). The combination of these two curves is a branch cut of $f(z)$. Each curve emanates from a point and goes to infinity. Each point is a *branch point.* They are roots of $z^2 + i = 0$. In other words, the branch point 0 of $\arg w$ induces branch points for $f(z)$. One can notice from Figure 2 that branch cut is where different branches are connected to each other to form "graph" of the multivalued function.

Next, we consider what happens if a different branch cut for $\arg w$ is used. Instead of restricting argument to be in $(-\pi, \pi]$, we restrict it to the interval $(\theta, \theta + 2\pi]$, where θ is some given number. In this case, the branch cut $\{\arg w = \theta \pmod{2\pi}\}$ is chosen. This choice affects how one calculates argument of a given point. If $\theta = -5\pi/7$ is chosen then the argument of $-1 - i$ will be $5\pi/4$ which belongs to $(-5\pi/7, -5\pi/7 + 2\pi)$. On the other hand, if $\theta = -\pi$ is chosen then the argument of $-1 - i$ will be $-3\pi/4$, which belongs to $(-\pi, -\pi + 2\pi]$. More generally, the branch of $\arg w$ created by branch cut $\{\arg w = \theta \pmod{2\pi}\}$ with range $(\theta, \theta + 2\pi]$ is given by:

$$\operatorname{Arg}_\theta(w) = \operatorname{Arg}(we^{-i(\theta+\pi)}) + \theta + \pi.$$

We define it in Mathematica as follows:

```
theta = -Pi/2
ARG[x_, y_] := Arg[(x + y*I)*E^(-I*(theta + Pi))] + theta + Pi
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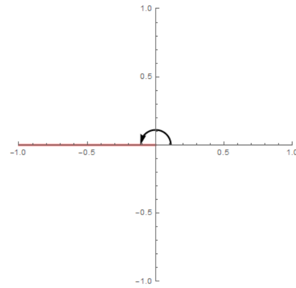
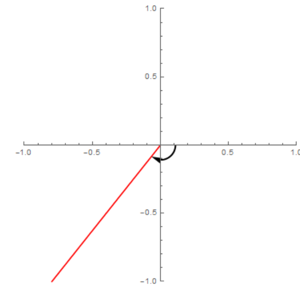


(a) $\theta = \pi$

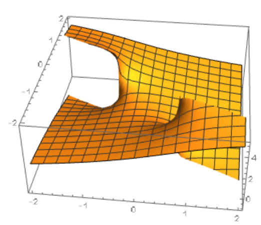


(b) $\theta = -5\pi/7$

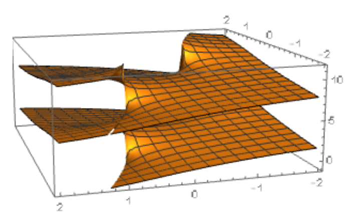
Figure 3: Different branch cuts of $\arg w$

In general, a different branch cut of $\arg w$ induces a different branch cut of $f(z)$. For example, when $\theta = -\pi/2$ then the branch cut of $f(z)$ is now made of two straight rays (Figure 4). In Mathematica,

```
q[k_] := Plot3D[ARG[Re[(x + y*I)^2 + I], Im[(x + y*I)^2 + I]] + k*2*Pi,
                        {x, -2, 2}, {y, -2, 2}]
Show[q[0], PlotRange -> All]
Show[q[0], q[1], PlotRange -> All]
```



(a) Branch $k = 0$



(b) Branches $k = 0$ and $k = 1$

Figure 4

When all branches are concatenated (i.e. when all $k \in \mathbb{Z}$ are put together), the graph of $f(z)$ is the same as before (Figure 1). Figure 1 corresponds to the case $\theta = -\pi$, in which the domain of each branch of $f(z)$ is $\mathbb{C}\backslash\{\text{two curves}\}$. Figure 4 corresponds to the case $\theta = -\pi/2$, in which the domain of each branch is $\mathbb{C}\backslash\{\text{two rays}\}$. In summary, different ways of choosing branch cut result in different ways of decomposing the graph of $f(z)$ into "floors". Each floor defines a continuous single-valued branch of $f(z)$.