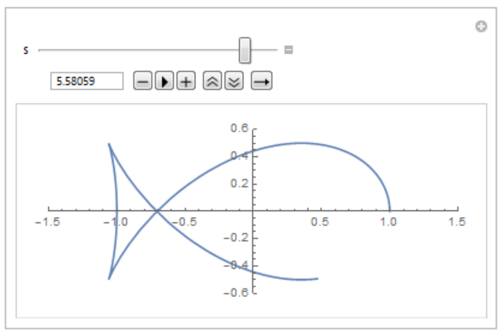
Complex integral via Mathematica

In this note, we will use Mathematica to

- compute complex integrals numerically,
- calculate work / flux of a vector field along / across a curve.

Consider a fish curve [1] with parametrization $\gamma(t) = \left(\cos t - \frac{1}{\sqrt{2}}\sin^2 t,\, \cos t \sin t\right)$ where $t \in [0, 2\pi]$ and complex function $f(z) = e^y + i\cos x$. To sketch γ in motion as t runs from 0 to 2π, run the following commands:

```
p[s_] := ParametricPlot[{Cos[t] - Sin[t]^2/Sqrt[2],
   Cos[t]*Sin[t]}, {t, 0, s}, PlotRange -> {{-1.5, 1.5}, {-.6, .6}}]
Manipulate[p[s], {s, .01, 2*Pi}]
```



Our first goal is to compute $\int_\gamma f(z)dz$. One can try several methods:

(a) *Use definition:*

$$\int_\gamma f(z)dz = \int_0^{2\pi} \left[e^{\cos t \sin t} + i\cos\left(\cos t - \frac{1}{\sqrt{2}}\sin^2 t\right)\right](\ldots)dt = \int_0^{2\pi} \ldots dt + i\int_0^{2\pi} \ldots dt$$

Each integral on the right hand side seems to be too complicated to compute analytically.

(b) *Use Fundamental Theorem of Calculus:*
It can be checked using Cauchy-Riemann equations that f is nowhere holomorphic. Thus, it has no antiderivatives.

(c) *Use Cauchy's Integral Formula:*
Since f is not in fraction form, Cauchy's Integral Formula is not applicable.

A practical approach is to compute the integral numerically based on the definition:

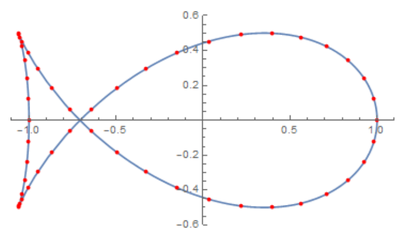
$$\int_\gamma f(z)dz = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(z_k)(z_{k+1} - z_k).$$

Here z_0, z_1,…,z_n are sample points on the path γ. One way to get such sample points is to sample the interval $t \in [0, 2\pi]$, for example by taking $t_0 = 0 < t_1 < t_2 < \ldots < t_n = 2\pi$ equally spaced. Then $z_k = \gamma(t_k)$ is a sample point on γ. In Mathematica:

```
n = 50
t[k_] := 2*Pi*k/n
z[k_] := Cos[t[k]] - Sin[t[k]]^2/Sqrt[2] + I*Cos[t[k]]*Sin[t[k]]
f[z_] := Exp[Im[z]] + I*Cos[Re[z]]
N[Sum[f[z[k]]*(z[k + 1] - z[k]), {k, 0, n - 1}]]
```

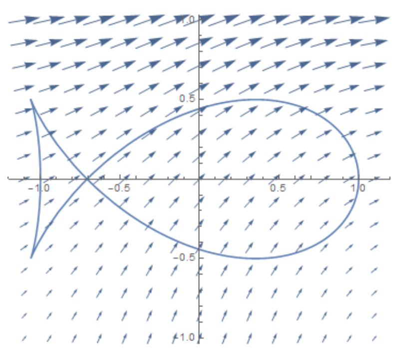
Increasing n for better approximations of the integral, we see that the integral is about -0.7906. Note that the plot of γ is $p[2\pi]$. One can plot the sample points on the fish curve by using the command **ListPlot** as follows.

```
samplePoints = Table[{Re[z[k]], Im[z[k]]}, {k, 0, n - 1}];
q = ListPlot[samplePoints, PlotStyle -> Red]
gamma = p[2*Pi]
Show[gamma, q, PlotRange -> {{-1.1, 1.1}, {-.6, .6}}]
```



The complex function $f(z) = e^y + i\cos x$ can be regarded as a vector field $f(x, y) = (e^y, \cos x)$. Our next goal is to compute the work done by f along γ, and the flux of f across γ. One can sketch the vector field together with curve on the same graph as follows.

```
vtfield = VectorPlot[{Exp[y], Cos[x]}, {x, -1.1, 1.1}, {y, -1, 1}]
Show[gamma, vtfield, PlotRange -> All]
```



Recall the formula

$$W[f, \gamma] + iF[f, \gamma] = \int_\gamma \bar{f}(z)dz$$

where $\bar{f}(z)$ is the Pólya vector field of f, given by $\bar{f}(z) = e^y - i\cos x$. One can repeat the numerical method mentioned earlier to compute $\int_\gamma \bar{f}(z)dz$, which turns out to be approximately -1.50099. Thus, $W[f, \gamma] \approx -1.50099$ and $F[f, \gamma] \approx 0$. Can you convince yourself of this result through the picture?

References

[1] http://mathworld.wolfram.com/FishCurve.html