Evaluating complex integral by series

We have learned how to compute complex integral $\int_\gamma f(z)dz$ numerically: by approximating it with Riemann sums. In this note, we consider another numerical method: using Taylor/Laurent series. The overall idea is rather simple: first, *decompose* the integrand $f(z)$ into simpler "modes", each of which is of the form z^k. Then compute integral at each mode. Then *synthesize* the results obtained at all modes to get the final result. The procedure is outlined as follows.

- Express the integrand as a power series (with positive and/or negative powers)

$$f(z) = \ldots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \ldots = \sum_{k=-\infty}^{\infty} a_k z^k.$$

- Do integration term by term

$$\int_\gamma f(z)dz = \ldots + a_{-2}\int_\gamma z^{-2}dz + a_{-1}\int_\gamma z^{-1}dz + a_0\int_\gamma 1dz + a_1\int_\gamma zdz + a_2\int_\gamma z^2dz + \ldots$$

Each integral on the right hand side is computable. For any integer $k \neq -1$, function z^k has antiderivative $\frac{z^{k+1}}{k+1}$ on $\mathbb{C}$ if $k \geq 0$, or on $\mathbb{C}\backslash\{0\}$ if $k \leq -2$. Thus,

$$\int_\gamma z^k dz = \frac{b^{k+1} - a^{k+1}}{k+1}$$

where a and b are the endpoints of γ. The case $k = -1$ needs more careful treatment. For example, if γ is not a simple loop, one can try to find a branch cut for the logarithm that avoids γ. Then

$$\int_\gamma z^{-1}dz = \mathcal{L}\text{og}\, b - \mathcal{L}\text{og}\, a.$$

If γ is a simple loop then

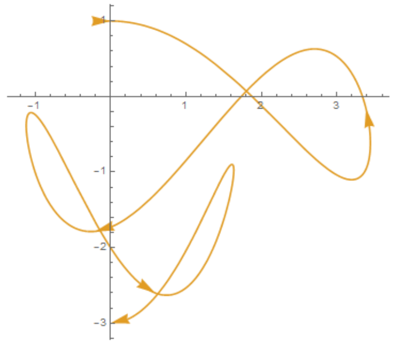
$$\int_\gamma z^{-1}dz = \pm 2\pi i$$

where the sign is determined by whether γ is positive or negatively oriented. To compare two numerical methods "Riemann sum" and "mode decomposition", one sees that the first method seemingly ignores the integrand, only investigating the path. The second method does the opposite. The second method seems to be more advantageous if the path parametrization is complicated whereas the integrand is simple.

Let us demonstrate the method through an example. Let γ be a curve with parametrization

$$\begin{cases} x(t) = \sin t + \sin 2t + 2\sin 3t, \\ y(t) = -1 + \cos t + \cos 7t \end{cases} \qquad t \in [0, \pi]$$

and $f(z) = \frac{\cos z}{z^3}$. In 1960s Robert Risch developed an algorithm to check if a function has an antiderivative that is an elementary function, i.e. a function that can be written in terms of power functions, trigonometric functions, exponential functions, logarithm,. . . or combinations among themselves using the four algebraic operations. It happens that antiderivative of function $\frac{\cos z}{z^3}$ is not an elementary function. It is more possible (and practical) to compute the integral numerically than analytically. We see that $f(z)$ is holomorphic on $\mathbb{C}\backslash\{0\}$. Thus, it possesses Laurent



series about 0. We decompose the function $f(z)$ as

$$f(z) = z^{-3} \cos z = z^{-3} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-3}}{(2k)!}.$$

Since γ lies in $\mathbb{C}\backslash\{0\}$, one can integrate the series term by term:

$$\int_\gamma f(z)dz = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int_\gamma z^{2k-3} dz.$$

If $k \neq 1$ then $2k - 3 \neq -1$ and

$$\int_\gamma z^{2k-3} dz = \frac{z^{2k-2}}{2k-2} \bigg|_i^{-3i} = \frac{(-3i)^{2k-2} - i^{2k-2}}{2k-2}.$$

If $k = 1$ then $2k - 3 = -1$. We see that γ does not intersect the negative real line. One can choose the principal logarithm as an antiderivative of z^{-1}.

$$\int_\gamma z^{-1} dz = \operatorname{Log} z \bigg|_i^{-3i} = \operatorname{Log}(-3i) - \operatorname{Log} i = \ln 3 - i\pi.$$

Now we combine the cases $k = 0$, $k = 1$ and $k \geq 2$ together:

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma z^{-3} dz - \frac{1}{2} \int_\gamma z^{-1} dz + \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k)!} \int_\gamma z^{2k-3} dz \\ &= \underbrace{\frac{z^{-2}}{-2} \bigg|_i^{-3i}}_{=-4/9} - \frac{1}{2}(\ln 3 - i\pi) + \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k)!} \frac{(-3i)^{2k-2} - i^{2k-2}}{2k-2}\end{aligned}$$

To compute numerically this series, one needs to truncate the series, say let k run from 2 to some large number n instead of ∞. In Mathematica,

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n = 50
-4/9 - 1/2*(Log[3] - I*Pi) +  N[Sum[(-1)^k/Factorial[2*k]
              *((-3*I)^(2*k - 2) - I^(2*k - 2))/(2*k - 2),
               {k, 2, n}]]
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We see that the integral is about $-1.19144 + 1.5708i$.