Some review problems for Midterm

1. Where are the following functions differentiable? Where are they holomorphic? Determine their derivatives at points where they are differentiable.

   (a) $x^2 + y^2 + i2xy$

   (b) $\bar{z}^2 + z$

   (c) $\frac{ix+1}{y}$

2. Find all real constants a and b such that $f(z) = (2x - y) + i(ax + by)$ is an entire function.

3. Determine and sketch the region of continuity of the following complex functions.

   (a) $\frac{z+1}{z^2+1}$

   (b) $\frac{x+iy}{|z|-1}$

   (c) $\sqrt{iz-1}$
   Hint: write $z = x + iy$

   (d) $\sqrt{z+1} + \sqrt{2z-i}$

4. Determine whether the following limits is a complex number, infinity or does not exist.

   (a) $$\lim_{z \to i} \frac{z}{z+1}$$

   (b) $$\lim_{z \to 0} \frac{|z|^2}{z}$$

   (c) $$\lim_{z \to \infty} z^2$$

   (d) $$\lim_{z \to \infty} \frac{z+i}{iz+1}$$

   (e) $$\lim_{z \to \infty} \frac{1}{z-a}$$
   where a is a given complex number.

5. Let $f(z) = \frac{z^2}{|z|^2}$

   (a) Find $\lim_{z \to 0} f(z)$ as $z \to 0$ along the line $y = x$.

   (b) Find $\lim_{z \to 0} f(z)$ as $z \to 0$ along the line $y = 2x$.

   (c) Find $\lim_{z \to 0} f(z)$ as $z \to 0$ along the parabola $y = x^2$.

   (d) Does the limit $\lim_{z \to 0} f(z)$ exist?

6. Consider a triangle with vertices at 1, $2 + 2i$, $3 - i$ oriented clockwise.

   (a) Draw the triangle and mark the orientation on its edges.

   (b) Find a parametrization for each of its edges.

7. Write the following complex numbers in standard form.

   (a) $e^{\frac{\pi}{3} - 2i}$

   (b) $\tan\left(\frac{\pi+2i}{4}\right)$

   (c) $\text{Log}(-\sqrt{3} - i)$

   (d) $(1 + i)^{2-i}$

8. Solve for all complex numbers z such that

   (a) $e^z = -\frac{i\pi}{3}$

   (b) $\cos z = 2$

   (c) $z^2 + (1 + i)z + 5i = 0$

   (d) $z^2 + \bar{z}^2 = i$

Answer key

1. (a) f is differentiable at all points on the real axis, and nowhere else. It is not holomorphic at any point. On the real axis, $f'(z) = f'(x + 0i) = 2x$.
   (b) f is differentiable at $z = 0$, and nowhere else. It is not holomorphic at any point. $f'(0) = 1$.
   (c) f is differentiable at $z = i$, and nowhere else. It is not holomorphic at any point. $f'(i) = i$.
2. $a = 1$, $b = 2$
3. (a) $\mathbb{C}\backslash\{\pm i\}$
   (b) The interior and exterior of the circle centered at the origin with radius 1.
   (c) $\mathbb{C}\backslash\{z : \ x = 0, \, y \geq -1\}$
   (d) $\mathbb{C}\backslash(\{z : \ x \leq -1, \, y = 0\} \cup \{z : \ x \leq 0, \ y = 1/2\})$
4. (a) $\frac{1}{2} + \frac{i}{2}$
   (b) 0
   (c) ∞
   (d) $-i$
   (e) 0
5. (a) i
   (b) $-\frac{3}{5} + \frac{4}{5}i$
   (c) 1
   (d) No
6. Note that parametrization of a straight path from z_1 to z_2 is $z(t) = z_1 + t(z_2 - z_1)$, $0 \leq t \leq 1$.
7. One can double-check results with Mathematica, for example

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ComplexExpand[(1 + I)^(2 - I)]
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8. (a) $\ln \frac{\pi}{3} + i(-\frac{\pi}{2} + k2\pi)$
   (b) $k2\pi - i \ln(2 \pm \sqrt{3})$
   (c) $1 - 2i$, $-2 + i$
   (d) No solutions

Formula provided on Midterm

$\log z = \ln |z| + i \arg z$

$z^a = e^{a \mathrm{Log}\, z}$ (a is non-integer)

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$\arcsin z = -i \log(iz + \sqrt{1 - z^2})$

$\arccos z = -i \log(z + i\sqrt{1 - z^2})$

$\arctan z = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)$

Cauchy–Riemann equations:

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$