MATH 483, MIDTERM EXAM, SPRING 2019

Name	Student ID

- Read the instruction of each problem carefully.
- The exam has 6 pages. **Circle your final results.**
- To get full credit for a problem **you must show your work**. Answers not supported by valid arguments will get little or no credit.

Problem	Possible points	Earned points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

Some helpful formula:

$$\log z = \ln|z| + i \arg z$$

$$z^a = e^{a \operatorname{Log} z} \ (a \text{ is non-integer})$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\arcsin z = -i \log(iz + \sqrt{1-z^2})$$

$$\arccos z = -i \log(z + i\sqrt{1-z^2})$$

$$\arctan z = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)$$

Cauchy–Riemann equations:

$$\begin{cases} \partial_x u &= \partial_y v \\ \partial_y u &= -\partial_x v \end{cases}$$

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Problem 1. Express the following numbers in complex standard form $a + ib$.

(a) (5 points) $(-1)^i$

$$(-1)^i = e^{i \operatorname{Log}(-1)}$$

$$\operatorname{Log}(-1) = \ln|1| + i\operatorname{Arg}(-1) = 0 + i\pi = i\pi$$

$$\rightsquigarrow (-1)^i = e^{i i \pi} = e^{-\pi}$$

(b) (5 points) $\sin(\frac{\pi}{2} + i)$

$$= \frac{e^{i(\frac{\pi}{2}+i)} - e^{-i(\frac{\pi}{2}+i)}}{2i} = \frac{e^{-1+i\frac{\pi}{2}} - e^{1-i\frac{\pi}{2}}}{2i}$$

$$= \frac{e^{-1}\operatorname{cis}(\frac{\pi}{2}) - e\operatorname{cis}(-\frac{\pi}{2})}{2i}$$

$$= \frac{e^{-1}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) - e\cos(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}))}{2i}$$

$$= \frac{e^{-1}(0+i) - e(0-i)}{2i}$$

$$= \frac{e^{-1}i + ei}{2i}$$

$$= \frac{e^{-1} + e}{2}$$

Problem 2. (10 points) Find all complex number z's such that $\cos z = 2$. Express the result in complex standard form.

$$z = \arccos 2$$

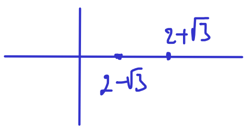
$$= -i \log\left(2 + i\sqrt{1-2^2}\right)$$

$$= -i \log\left(2 + i\sqrt{-3}\right) = -i \log\left(2 \pm i\, i\sqrt{3}\right) = -i \log\left(2 \pm \sqrt{3}\right)$$

If plus sign is taken,

$$z = -i\left[\ln(2+\sqrt{3}) + i \arg(2+\sqrt{3})\right]$$

$$= -i\left[\ln(2+\sqrt{3}) + i k 2\pi\right]$$



$$= k2\pi - i \ln(2+\sqrt{3})$$

If minus sign is taken,

$$z = -i\left[\ln(2-\sqrt{3}) + i \arg(2-\sqrt{3})\right]$$

$$= -i\left[\ln(2-\sqrt{3}) + i k 2\pi\right]$$

$$= k2\pi - i \ln(2-\sqrt{3})$$

Conclusion: $z = k2\pi - i\ln(2 \pm \sqrt{3})$

Problem 3. (10 points) Find all complex roots of the equation $z^2 + (1 - 2i)z - 3 - i = 0$. Express the results in complex standard form.

$$\Delta = (1-2i)^2 - 4(-3-i) = 1 - 4i + 4i^2 + 12 + 4i$$

$$= 9$$

$$\sqrt{\Delta} = \pm 3$$

$$z_{1,2} = \frac{-(1-2i) \pm \sqrt{\Delta}}{2} = \frac{-1 + 2i \pm 3}{2}$$

$$z_1 = \frac{-1 + 2i + 3}{2} = 1 + i$$

$$z_2 = \frac{-1 + 2i - 3}{2} = -2 + i$$

Problem 4. Put $z_1 = -i$ and $z_2 = 2 + i$.

(a) (3 points) Write the equation of the line segment from z_1 to z_2 in complex standard form $z(t) = x(t) + iy(t)$. Call this path γ_1.

$$z(t) = z_1 + t(z_2 - z_1) = -i + t(2+i-(-i)) = -i + t(2+2i)$$

$$z(t) = 2t + (2t-1)i \qquad 0 \le t \le 1.$$

(b) (2 points) Verify that γ_1 passes through point $z = 1$.

Plug $t = \frac{1}{2}$ into the equation of γ_1 above:

$$z\left(\tfrac{1}{2}\right) = 2 \cdot \frac{1}{2} + \left(2 \cdot \tfrac{1}{2} - 1\right) i = 1$$

Thus, 1 lies on γ_1.

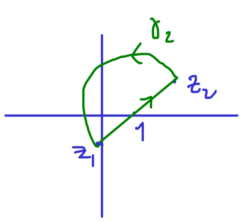
(c) (2 points) Verify that $z = 1$ is the midpoint of γ_1.

$$|1 - z_1| = |1 - (-i)| = |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$|1 - z_2| = |1 - (2+i)| = |-1 - i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Thus, 1 is the midpoint of γ_1.

(d) (3 points) Let γ_2 be the upper half of the circle centered at 1 passing through z_1. It starts at z_2 and ends at z_1. Write the equation for γ_2 in complex standard form.



The radius of the circle is $|1 - z_1| = \sqrt{2}$.

Equation of γ_2: $z(t) = 1 + \sqrt{2}\, e^{it}$

$$= 1 + \sqrt{2}\cos t + i\sqrt{2}\sin t.$$

The range of t is $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Problem 5. Consider $f(z) = \frac{\sin z}{z}$.

(a) (2 points) Express $f(z)$ in complex standard form $f = u + iv$ when z lies on the real axis.

$z = x$

$$f(z) = \frac{\sin x}{x}$$

$$u(x,y) = \frac{\sin x}{x}$$

$$v(x,y) = 0$$

(b) (2 points) Find the limit of $f(z)$ as $z \to \infty$ along the x-axis toward the positive direction.

$$\lim_{x\to\infty} u(x,y) = \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

because $0 \le \left|\frac{\sin x}{x}\right| = \frac{|\sin x|}{|x|} \le \frac{1}{|x|} \to 0$

$\lim_{x\to\infty} v(x,y) = 0$. Therefore, $f(z) \to 0 + i0 = 0$ as $z \to \infty$ along x-axis.

(c) (2 points) Express $f(z)$ in complex standard form $f = u + iv$ when z lies on the imaginary axis.

$z = iy$

$$f(z) = \frac{\sin iy}{iy} = \frac{\frac{e^{iiy} - e^{-iiy}}{2i}}{iy} = \frac{e^{-y} - e^{y}}{2i^2 y} = \frac{e^{y} - e^{-y}}{2y}$$

$$u(x,y) = \frac{e^{y} - e^{-y}}{2y}$$

$$v(x,y) = 0$$

(d) (2 points) Find the limit of $f(z)$ as $z \to \infty$ along the y-axis toward the positive direction.

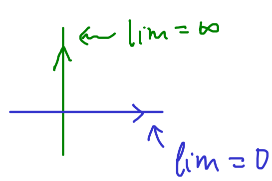
$$\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} \frac{e^{y} - e^{-y}}{2y} = \infty$$

because the e^y grows faster than y and $e^{-y} \to 0$.

$\lim_{y\to\infty} v(x,y) = 0$. Thus, $f(z) \to \infty$ as $z \to \infty$ along y-axis.

(e) (2 points) Find $\lim_{z\to\infty} f(z)$.

$\lim_{z\to\infty} f(z)$ DNE because limits along different paths are different.



Problem 6. (10 points) Where is the following function differentiable? Where is it holomorphic? Determine its derivative at points where it is differentiable.

$$f(z) = x^2y + x + i(xy^2 - x + y)$$

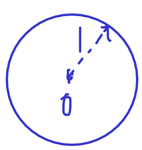
$$u(x,y) = x^2y + x$$

$$v(x,y) = xy^2 - x + y$$

$$\left.\begin{array}{ll} \partial_x u = 2xy+1, & \partial_x v = y^2-1 \\ \partial_y u = x^2, & \partial_y v = 2xy+1 \end{array}\right\}$$ These are continuous everywhere.

Cauchy-Riemann equations

$$\begin{cases} 2xy+1 = 2xy+1 \\ x^2 = -(y^2-1) \end{cases} \rightsquigarrow x^2+y^2=1.$$



The function $f(z)$ is differentiable at every point on the unit circle, and nowhere else. It is not holomorphic anywhere.

On the unit circle,

$$f'(z) = \partial_x u + i\partial_x v = 2xy + 1 + i(y^2-1).$$