Worksheet

5/3/2019

1. Where is the following function differentiable? Where is it holomorphic? Determine its derivative at points where it is differentiable.

$$f(z) = x^2 + y^2 + i2xy$$

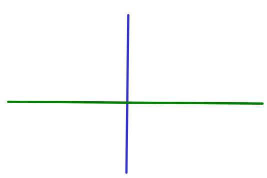
$$u = x^2 + y^2 \rightsquigarrow \partial_x u = 2x, \quad \partial_y u = 2y$$

$$v = 2xy \rightsquigarrow \partial_x v = 2y, \quad \partial_y v = 2x$$

These derivatives are continuous everywhere.

C-R eqs: $\begin{cases} 2x = 2x \\ 2y = -2y \end{cases} \rightsquigarrow \begin{cases} x \text{ arbitrary} \\ y = 0 \end{cases}$

$\rightsquigarrow$ f is differentiable everywhere on the real axis, and nowhere else



f is not holomorphic anywhere.

For $z = x + i0$,

$$f'(z) = \partial_x u(x,0) + i\,\partial_x v(x,0) = 2x + i \cdot 0 = 2x$$

2. Find all real constants a and b such that $f(z) = (2x - y) + i(ax + by)$ is an entire function.

$$u = 2x - y \rightsquigarrow \partial_x u = 2, \quad \partial_y u = -1$$

$$v = ax + by \rightsquigarrow \underbrace{\partial_x v = a, \quad \partial_y v = b}_{\text{These are constant functions, thus continuous}}$$

C-R eqs: $\begin{cases} 2 = b \\ -1 = -a \end{cases} \rightsquigarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$

3. Determine and sketch the region of continuity of the following complex functions.

(a) $\frac{z+1}{z^2+1}$

$z+1$ and z^2+1 are continuous everywhere.

The quotient $\frac{z+1}{z^2+1}$ is continuous everywhere except at z's such that $z^2+1=0$. Conclusion: $\mathbb{C}\setminus\{\pm i\}$

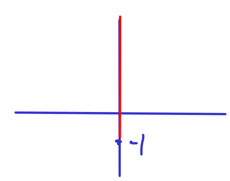
(b) $\sqrt{iz-1}$

Hint: write $z = x + iy$

$\sqrt{iz-1} = e^{\frac{1}{2}\log(iz-1)}$ is continuous everywhere except at z's such that $iz-1 \in \mathbb{R}_{\leq 0}$.

Write $iz-1 = i(x+iy)-1 = -y-1+ix \in \mathbb{R}_{\leq 0}$.

Thus, $\begin{cases} -y-1 \leq 0 \\ x = 0 \end{cases} \rightsquigarrow \begin{cases} y \geq -1 \\ x = 0 \end{cases}$



(c) $\sqrt{z+1} + \sqrt{2z-i}$

$\sqrt{z+1}$ is continuous everywhere except at z's such that $z+1 \in \mathbb{R}_{\leq 0}$.

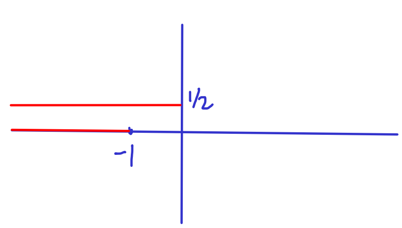
Write $z+1 = x+1+iy \in \mathbb{R}_{\leq 0}$

$\rightsquigarrow \begin{cases} x+1 \leq 0 \\ y = 0 \end{cases} \rightsquigarrow \begin{cases} x \leq -1 \\ y = 0 \end{cases}$

$\sqrt{2z-i}$ " " " $2z-i \in \mathbb{R}_{\leq 0}$

Write $2z-i = 2(x+iy)-i = 2x + i(2y-1) \in \mathbb{R}_{\leq 0}$

$\rightsquigarrow \begin{cases} 2x \leq 0 \\ 2y-1 = 0 \end{cases} \rightsquigarrow \begin{cases} x \leq 0 \\ y = \frac{1}{2} \end{cases}$



Conclusion: $\mathbb{C}$ minus the two red lines.

4. Determine whether the following limits is a complex number, infinity or does not exist.

(a)

$$\lim_{z\to\infty} \frac{z+i}{iz+1}$$

Hint: Divide numerator and denominator by z.

Use the fact that $\lim_{z\to\infty} \frac{1}{z} = 0$.

$$\lim_{z\to\infty} \frac{1+\frac{i}{z}}{i+\frac{1}{z}} = \frac{1+i\cdot 0}{i+0} = \frac{1}{i} = -i$$

(b)

$$\lim_{z\to\infty} \frac{1}{z-a}$$

where a is a given complex number.

$$\lim_{z\to\infty} \frac{\frac{1}{z}}{1-\frac{a}{z}} = \frac{0}{1-a\cdot 0} = 0$$

(c)

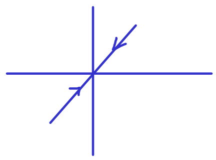
$$\lim_{z\to 0} \frac{|z|^2}{z}$$

$$\left|\frac{|z|^2}{z} - 0\right| = \left|\frac{|z|^2}{z}\right| = \frac{|z|^2}{|z|} = |z| \longrightarrow 0 \quad \text{as} \quad z\to 0$$

Thus, $\lim_{z\to 0} \frac{|z|^2}{z} = 0$

5. Let $f(z) = \frac{z^2}{|z|^2}$

(a) Find $\lim_{z \to 0} f(z)$ as $z \to 0$ along the line $y = x$.

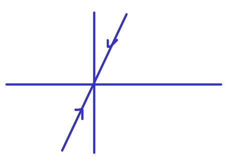


$z = t + it$

$$f(z) = \frac{(t+it)^2}{|t+it|^2} = \frac{t^2(1+i)^2}{t^2+t^2} = \frac{t^2(1+2i+i^2)}{2t^2} = i$$

Limit of $f(z)$ as $z \to 0$ along $y = x$ is i

(b) Find $\lim_{z \to 0} f(z)$ as $z \to 0$ along the line $y = 2x$.



$z = t + i2t$

$$f(z) = \frac{(t+i2t)^2}{t^2+(2t)^2} = \frac{t^2(1+2i)^2}{5t^2} = \frac{1+4i+4i^2}{5} = \frac{-3}{5} + \frac{4}{5}i$$

Limit of $f(z)$ as $z \to 0$ along $y=2x$ is $-\frac{3}{5} + i\frac{4}{5}$.

(c) Does the limit $\lim_{z \to 0} f(z)$ exist?

No, because limits of $f(z)$ as $z \to 0$ along different paths are not the same : $i \neq -\frac{3}{5} + i\frac{4}{5}$.