Worksheet

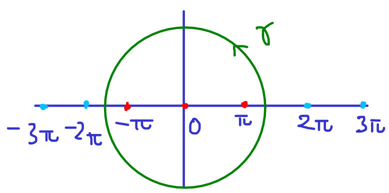
6/7/2019

1. Consider function $f(z) = \frac{\mathrm{Log}(z+5)}{\sin z}$.

   (a) Determine all singular point(s) of f enclosed in the circle $C_4(0)$. Are they isolated singularities?

Note that the roots of $\sin z = 0$ always lie on the real axis.

$\sin z = 0$ when $z = k\pi$ where $k \in \mathbb{Z}$. The only roots of $\sin z$ enclosed by $\gamma = C_4(0)$ are $z = 0, \pm\pi$.



Function f also has nonisolated singularities (on the line $\mathbb{R}_{\leq -5}$), but these points lie outside of γ.

Conclusion: $0, \pm\pi$ are the only sing. of f inside γ. They are isolated sing.

   (b) Which kind of isolated singularity are they (removable, pole, essential)? If they are poles, determine their orders.

When z is near to 0, $\mathrm{Log}(z+5)$ is near to $\ln 5$, and

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)$$

Thus,
$$f(z) \approx \frac{\mathrm{Log}\, 5}{z\left(1 - \frac{z^2}{3!} + \cdots\right)} = z^{-1} \underbrace{\frac{\mathrm{Log}\, 5}{1 - \frac{z^2}{3!} + \cdots}}_{\neq 0 \text{ when } z = 0}$$

We guess that 0 is a pole of order 1. To verify our guess, we compute

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z\,\mathrm{Log}(z+5)}{\sin z} = \ln 5 \lim_{z\to 0} \frac{z}{\sin z} \overset{L'H}{=} \ln 5 \lim_{z\to 0} \frac{1}{\cos z}$$

$$= \ln 5 \neq 0, \neq \infty$$

Similarly, $\pm\pi$ are also poles of order 1 because

$$\lim_{z\to \pi} (z-\pi) f(z) = \lim_{z\to\pi} (z-\pi)\frac{\mathrm{Log}(z+5)}{\sin z} = \ln(\pi+5) \lim_{z\to 0} \frac{z-\pi}{\sin z} \overset{L'H}{=} -\ln(5+\pi) \neq 0, \neq \infty$$

$$\lim_{z\to -\pi} (z+\pi) f(z) = \cdots$$

(c) Compute the residue of f at each of these singularities.

use the formula $\operatorname{Res}[f, z_0] = a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]$

Here $n = 1$ for $z_0 = 0, \pm\pi$.

$$\operatorname{Res}[f, 0] = \frac{1}{0!} \lim_{z \to 0} \frac{d^0}{dz^0} \left[(z-0)^1 f(z) \right] = \lim_{z \to 0} z f(z)$$

$$= \ln 5 \text{ (as computed above)}$$

$$\operatorname{Res}[f, \pi] = \cdots = \lim_{z \to \pi} (z-\pi) f(z) = -\ln(5+\pi) \text{ (as computed above)}$$

$$\operatorname{Res}[f, -\pi] = \cdots = \lim_{z \to -\pi} (z+\pi) f(z) = -\ln(5-\pi)$$

(d) Evaluate the integral $\int_\gamma f(z)dz$ where γ is the circle $C_4(0)$ oriented counterclockwise.

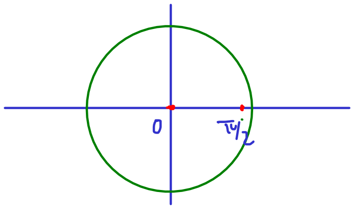
By Cauchy's Residue theorem,

$$\int_\gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}[f; 0] + \operatorname{Res}[f; \pi] + \operatorname{Res}[f, -\pi] \right)$$

$$= 2\pi i \left(\ln 5 - \ln(5+\pi) - \ln(5-\pi) \right)$$

$$= 2\pi i \ln \frac{5}{25-\pi^2}$$

2. Compute $\int_\gamma \frac{z+1}{(z-\frac{\pi}{2})^2 \sin z} dz$ where γ is the circle $C_2(0)$ oriented counterclockwise.



The only singularities of $f(z) = \dfrac{z+1}{(z-\frac{\pi}{2})^2 \sin z}$ that lie in γ are $z=0$ and $z=\frac{\pi}{2}$.

We see that
$$f(z) = \left(z-\frac{\pi}{2}\right)^{-2} \underbrace{\frac{z+1}{\sin z}}_{\neq 0 \text{ when } z=\frac{\pi}{2}}$$

and
$$f(z) = \frac{z+1}{(z-\frac{\pi}{2})^2 \left(z - \frac{z^3}{6} + \cdots\right)} = \frac{z+1}{(z-\frac{\pi}{2})^2 z \left(1 - \frac{z^2}{6} + \cdots\right)}$$
$$= z^{-1} \underbrace{\frac{z+1}{(z-\frac{\pi}{2})^2 \left(1 - \frac{z^2}{6} + \cdots\right)}}_{\neq 0 \text{ when } z=0}$$

Thus, we guess that $z=\frac{\pi}{2}$ is a pole of order 2, $z=0$ is a pole of order 1.

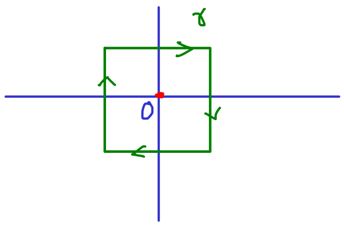
Verify:
$$\lim_{z\to\frac{\pi}{2}} \left(z-\frac{\pi}{2}\right)^2 f(z) = \lim_{z\to\pi/2} \frac{z+1}{\sin z} = \frac{\pi/2+1}{1} \neq 0, \neq \infty$$
$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{(z+1)z}{(z-\frac{\pi}{2})^2 \sin z} = \frac{1}{(-\frac{\pi}{2})^2} \lim_{z\to 0} \frac{z}{\sin z} \overset{L'H}{=} \frac{4}{\pi^2}$$
$\neq 0, \neq \infty$

Next,
$$\operatorname{Res}\left[f, \frac{\pi}{2}\right] = \frac{1}{1!} \lim_{z\to\frac{\pi}{2}} \frac{d}{dz}\left[\left(z-\frac{\pi}{2}\right)^2 f(z)\right] = \lim_{z\to\frac{\pi}{2}} \frac{d}{dz}\left(\frac{z+1}{\sin z}\right)$$
$$= \lim_{z\to\frac{\pi}{2}} \frac{1 \sin z - (z+1)\cos z}{\sin^2 z} = 1$$

$$\operatorname{Res}[f, 0] = \frac{1}{0!} \lim_{z\to 0} z f(z) = \frac{4}{\pi^2} \text{ (as computed above).}$$

By Cauchy's Residue theorem,
$$\int_\gamma f(z)dz = 2\pi i \left(\operatorname{Res}\left[f, \frac{\pi}{2}\right] + \operatorname{Res}[f,0]\right)$$
$$= 2\pi i \left(1 + \frac{4}{\pi^2}\right)$$

3. Compute $\int_\gamma z^2 \sin(\frac{1}{z})dz$ where γ is the boundary of square with vertices at $\pm 1 \pm i$ negatively oriented.



The only posible singularity of $f(z) = z^2 \sin\left(\frac{1}{z}\right)$ is $z=0$. The function is holomorphic everywhere else. Thus, $f(z)$ can be written as a Laurent series around 0.

Let's try to write a Laurent series of $f(z)$:

$$\sin w = w - \frac{w^3}{3!} + \frac{w^5}{5!} - \dots$$

Now substitute $w = \frac{1}{z} = z^{-1}$:

$$\sin \frac{1}{z} = z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \dots$$

Thus,

$$f(z) = z^2 \sin \frac{1}{z} = z - \frac{z^{-1}}{3!} + \frac{z^{-3}}{5!} - \dots$$

This is the Laurent series of $f(z)$ around 0. (And we see that 0 is an essential singularity.)

$$\operatorname{Res}[f;0] = \text{coefficient of } z^{-1} = -\frac{1}{6}$$

By Cauchy's Residue theorem,

$$\int_\gamma f(z)dz = -\int_{\underbrace{\gamma_{Rev}}} f(z)\,dz = -2\pi i \operatorname{Res}[f;0] = -2\pi i\left(-\frac{1}{6}\right) = \frac{\pi i}{3}.$$

positively oriented loop