Guessing region of continuity and visualizing multivalued functions

In this note, we will use Mathematica to

- Guess the region of continuity of a (single-valued) complex function.
- Visualize a multivalued function.

1 Guess the region of continuity

Let us consider the function $f(z) = \mathrm{Log}(z^2+1)$. *We want to know the region of z where f is continuous.* By the definition of the principal logarithm,

$$f(z) = \ln|z^2+1| + i\mathrm{Arg}(z^2+1).$$

The real part of $f(z)$ is $u(z) = \ln|z^2+1|$. The imaginary of $f(z)$ is $v(z) = \mathrm{Arg}(z^2+1)$. For f to be continuous at z, both u and v have to be continuous at z. Intuitively, one can view function u as a real-valued function of two real variables

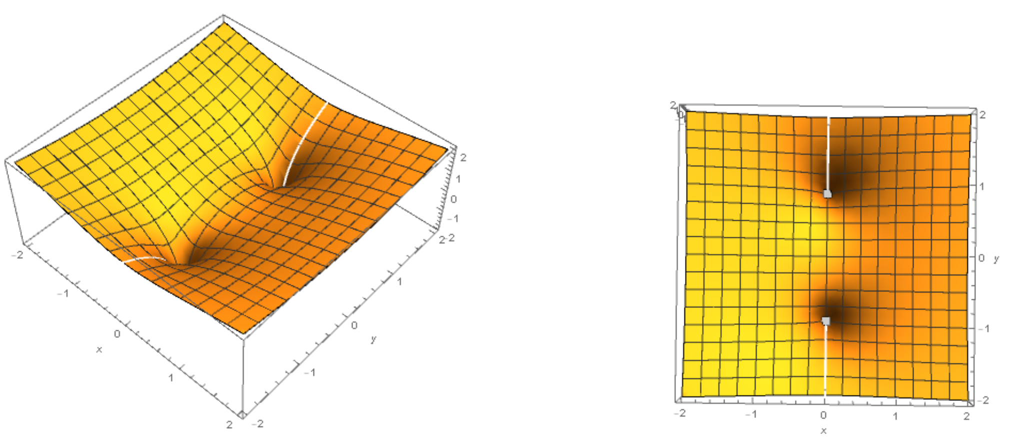
$$u(x,y) = \ln|(x+iy)^2+1| = \ln|x^2-y^2+1+i2xy| = \ln\sqrt{(x^2-y^2+1)^2+(2xy)^2}.$$

One can similarly view v as a real-valued function of two real variables. We can define u and v concisely in Mathematica by:

```
f[z_] := Log[z^2 + 1]
u[z_] := Re[f[z]]
v[z_] := Im[f[z]]
```

One can draw the graphs of u by (Figure 1)

```
Plot3D[u[x + I*y], {x, -2, 2}, {y, -2, 2}, AxesLabel -> Automatic]
```



(a) Looking from an angle

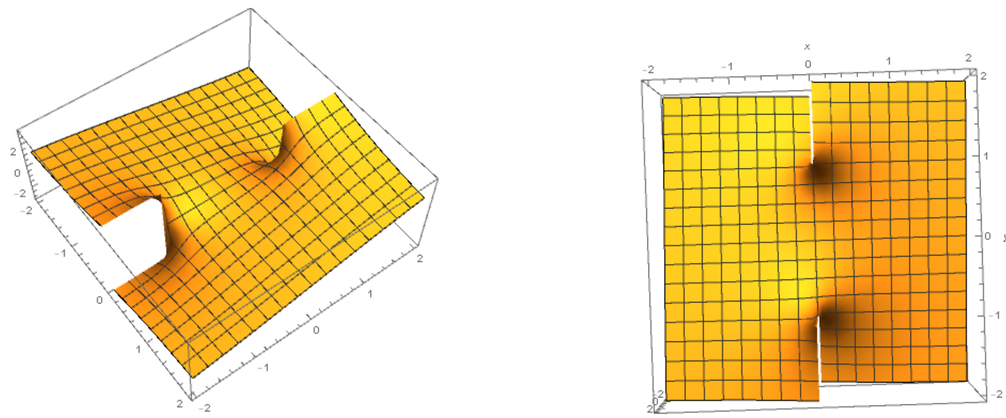
(b) Looking from above

Figure 1: Graph of u

From the picture, we can guess that u is continuous everywhere except at two points $x = 0$, $y = 1$ and $x = 0$, $y = -1$. The graph of u at each of these two points look like an infinite well, indicating that the limit of u is equal to ∞ at these points. One may also notice two slits on the graph. They are insignificant because u does not behave abnormally (discontinuously) across each slit.

One can draw the graphs of v by (Figure 2)

```
Plot3D[v[x + I*y], {x, -2, 2}, {y, -2, 2}, AxesLabel -> Automatic]
```



(a) Looking from an angle

(b) Looking from above

Figure 2: Graph of v

From the picture, we can guess that v is continuous everywhere except on two lines $x = 0$, $y \geq 1$ and $x = 0$, $y \leq -1$. The values of $v(z)$ jump across these lines. From here, we can guess that the function $f(z) = u(z) + iv(z)$ is continuous everywhere except on the line $x = 0$, $y \geq 1$ and the line $x = 0$, $y \leq -1$. An analytic justification is needed to confirm this observation. However, we skip this step here.

2 Visualize multivalued functions

Let us consider $g(z) = \log(z^2 + 1)$. By the definition of logarithm,

$$g(z) = \ln|z^2 + 1| + i\arg(z^2 + 1).$$

This is a multivalued function because arg is a multivalued function. The real part of $g(z)$ is $u(z) = \ln|z^2 + 1|$, which is a single-valued function. We plotted the graph of u above. The imaginary part of $g(z)$ is

$$w(z) = \arg(z^2 + 1) = \mathrm{Arg}(z^2 + 1) + k2\pi = v(z) + k2\pi,$$

which is a multivalued function.

If $k = 0$ then $w(z) = v(z)$, whose graph is given in Figure 2. This is one "branch" of $w(z)$. Other branches are obtained by choosing different integer values of k. For each k, one can draw the graph of $v(z) + k2\pi$. The combination of *all of these branches* gives a full picture of $w(z)$. In other words, the "graph" of the multivalued function $w(z)$ is a concatenation of copies of the graph of $v(z)$. Each copy is a vertical shift by a multiple of 2π of the graph of $v(z)$. In Mathematica,

```
p[k_] := Plot3D[v[x+I*y] + k*2*Pi, {x, -2, 2}, {y, -2, 2}]
Show[p[0], p[1], PlotRange -> All]
```

Figure 3 shows two branches of function $w(z)$. To see three branches, one can write (Figure 4)

```
Show[p[0], p[1], p[2], PlotRange -> All]
```

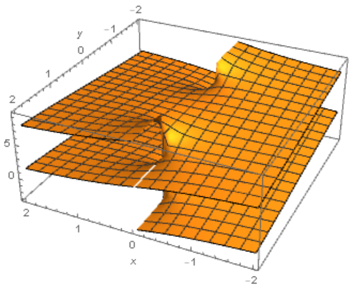


Figure 3: Two branches of multivalued function $w(z)$

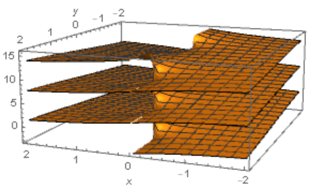


Figure 4: Three branches of multivalued function $w(z)$

The lines where v is discontinuous are the places where a new branch is glued to. The concatenation of all of the branches forms a surface (smooth everywhere) that looks like a parking deck. This is an example of *Riemann surface*.