MTH 483/583 Complex Variables – Homework 2

Solution Key

Spring 2020

Exercise 1. Find all complex roots of the following cubic equation. Write them in standard form $z = a + ib$ where a and b are numerical values (round to 4 digits after decimal point).

(a) $z^3 + 3z + 1 = 0$

Answer: From the Cardano's formula (see also the lecture note for roots of cubic polynomial), we split z into the sum of u, v: $z = u + v$. Then note that

$$\triangle = q^2 + \frac{4p^3}{27} = 1 + \frac{4}{27}27 = 5$$

and so

$$u = \left[\frac{\sqrt{5}-1}{2}\right]^{1/3}$$

.

The third roots of $\frac{\sqrt{5}-1}{2}$ are

$$\Gamma_1 := \left\{ \left(\frac{\sqrt{5}-1}{2}\right)^{1/3} e^{i\left(\frac{2k\pi}{3}\right)} : k = 0, 1, 2 \right\}$$

Similarly,

$$v = \left[\frac{-\sqrt{5}-1}{2}\right]^{1/3}$$

and the third roots of $\frac{-\sqrt{5}-1}{2}$ are

$$\Gamma_2 := \left\{ \left(\frac{\sqrt{5}+1}{2}\right)^{1/3} e^{i\left(\frac{-\pi+2m\pi}{3}\right)} : m = 0, 1, 2 \right\}$$

Let

$$u_1 = \left(\frac{\sqrt{5}-1}{2}\right)^{1/3}, \; u_2 = \left(\frac{\sqrt{5}-1}{2}\right)^{1/3} e^{i\frac{2\pi}{3}}, \; u_3 = \left(\frac{\sqrt{5}-1}{2}\right)^{1/3} e^{i\frac{4\pi}{3}}$$

and

$$v_1 = \left(\frac{\sqrt{5}+1}{2}\right)^{1/3} e^{-i\frac{\pi}{3}},\ v_2 = \left(\frac{\sqrt{5}+1}{2}\right)^{1/3} e^{i\frac{\pi}{3}},\ v_3 = \left(\frac{\sqrt{5}+1}{2}\right)^{1/3} e^{i\pi}$$

We choose the pairs of u and v such that $uv = -p/3 = -1$. A quick way to do this is to choose the pairs of u and v such that $\arg u + \arg v = \arg(-1) = \pi$ (in modulo 2π). These pairs are (u_1, v_3), (u_2, v_2), (u_3, v_1). Therefore, the three complex roots of the given equation are

$$z_1 = u_1 + v_3 \approx \boxed{-0.3222}$$

$$z_2 = u_2 + v_2 \approx \boxed{0.1611 + i1.7544}$$

$$z_3 = u_3 + v_1 \approx \boxed{0.1611 - i1.7544}$$

$\square$

You can double-check your results with Mathematica by the command NSolve:

```
NSolve[z^3 + 3*z + 1 == 0, z]
```

(b) $2z^3 - 6z^2 + 2z + 1 = 0$

Answer: Set $a = 2$, $b = -6$, $c = 2$, $d = 1$.

Divide by the leading coefficient:

$$z^3 - 3z^2 + z + \frac{1}{2} = 0 \tag{1}$$

and then use the change of variables: $x := z + \frac{b}{3a} = z - 1$, or equivalently, $z = x + 1$. Substitute this into the equation (1):

$$(x+1)^3 - 3(x+1)^2 + (x+1) + \frac{1}{2} = 0$$

multiply them out and simplify to obtain

$$x^3 - 2x - \frac{1}{2} = 0$$

Use the same argument as we did in part (a), we find that

$$\triangle = -\frac{101}{108} = \frac{101}{108} i^2 \implies \sqrt{\triangle} = \pm\sqrt{\frac{101}{108}} i$$

so

$$u = \left(\frac{1}{4} + i\sqrt{\frac{101}{432}}\right)^{1/3}, \quad v = \left(\frac{1}{4} - i\sqrt{\frac{101}{432}}\right)^{1/3}$$

we get

$$u_k = \left(\frac{2\sqrt{6}}{9}\right)^{1/3} e^{i(\theta_1+2k\pi)/3}, \quad k = 0, 1, 2$$

where $\theta_1 = \arctan(\sqrt{101/27})$

$$v_k = \left(\frac{2\sqrt{6}}{9}\right)^{1/3} e^{i(\theta_2+2k\pi)/3}, \quad k = 0, 1, 2$$

where $\theta_2 = \arctan(-\sqrt{101/27}) = -\theta_1$.

We choose the pairs of u and v such that $uv = -p/3 = 2/3$. A quick way to do this is to choose the pairs of u and v such that $\arg u + \arg v = \arg(2/3) = 0$ (in modulo 2π). These pairs are (u_1, v_3), (u_2, v_2), (u_3, v_1). Therefore, the three complex roots of the given equation are (u_1, v_1), (u_2, v_2), (u_3, v_3). Therefore, the three values of x are

$$x_1 = u_1 + v_1 \approx 1.5257$$

$$x_2 = u_2 + v_2 \approx -1.2670$$

$$x_3 = u_3 + v_3 \approx -0.2587$$

Therefore, the three roots of the given equation are

$$z_1 = u_1 + v_1 \approx \boxed{2.5257}$$

$$z_2 = u_2 + v_2 \approx \boxed{-0.2670}$$

$$z_3 = u_3 + v_3 \approx \boxed{0.7413}$$

□

Exercise 2. Express the following complex numbers in either standard form or polar form.

(a) $e^{e^{1+2i}} = e^{e(\cos(2)+i\sin(2))} = e^{e\cos(2)}e^{ie\sin(2)} = \boxed{e^{e\cos(2)}\cos(e\sin(2)) + ie^{e\cos(2)}\sin(e\sin(2))}$.

(b) $e^{|2-i|} = \boxed{e^{\sqrt{5}}}$.

(c) $\sin(-1+i) = \frac{e^{i(-1+i)}-e^{-i(-1+i)}}{2i} = \frac{e^{-1}e^{-i}-ee^{i}}{2i} = \frac{e^{-1}(\cos 1 - i\sin 1)-e(\cos 1+i\sin 1)}{2i}$

$= \frac{(e^{-1}-e)\cos 1 + i(-e^{-1}-e)\sin 1}{2i} = \boxed{\frac{-(e+e^{-1})\sin 1}{2} + i\frac{(e-e^{-1})\cos 1}{2}}$

(d) $\tan(i) = \frac{e^{i(i)}-e^{-i(i)}}{i(e^{i(i)}+e^{-i(i)})} = \boxed{\left(\frac{e-e^{-1}}{e+e^{-1}}\right)i}$

Exercise 3. Recall de Moivre's formula:

$$(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$$

for any real number x and integer number n. Apply this formula to express $\cos(5x)$ and $\sin(5x)$ in terms of $\cos x$ and $\sin x$

Proof. Note that

$$\begin{aligned}(\cos x - i \sin x)^5 &= (\cos(-x) + i \sin(-x))^5 = \cos(-5x) + i \sin(-5x) \\ &= \cos(5x) - i \sin(5x) \end{aligned} \tag{2}$$

On the other hand, de Moivre's formula gives us:

$$(\cos x + i \sin x)^5 = \cos(5x) + i \sin(5x) \tag{3}$$

Combine (3) and (3), we have

$$\cos(5x) = \frac{(\cos x + i \sin x)^5 + (\cos x - i \sin x)^5}{2}$$

and

$$\sin(5x) = \frac{(\cos x + i \sin x)^5 - (\cos x - i \sin x)^5}{2i}$$

□

Exercise 4. Recall that the sine and cosine functions are defined in terms of the exponential function. Use the identity $e^{u+v} = e^u e^v$ for any $u, v \in \mathbb{C}$ to prove the following identity:

$$\sin(z + w) = \sin z \cos w + \cos z \sin w \quad \forall z, w \in \mathbb{C}$$

Proof. Recall that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

So

$$\sin z \cos w + \cos z \sin w = \frac{e^{iz} - e^{-iz}}{2i} \frac{e^{iw} + e^{-iw}}{2} + \frac{e^{iz} + e^{-iz}}{2} \frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{e^{i(z+w)} + e^{i(z-w)} - e^{-i(z-w)} - e^{-i(z+w)} + e^{i(z+w)} - e^{i(z-w)} + e^{-i(z-w)} - e^{-i(z+w)}}{4i}$$

$$= \frac{2e^{i(z+w)} - 2e^{-i(z+w)}}{4i} = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z + w)$$

□

Exercise 5. For $z, w \in \mathbb{C}$, show the following identities.
We let $z = a + ib$ and $w = c + id$ in the following.

(a) $\overline{z+w} = \bar{z} + \bar{w}$:

Proof. Since $z + w = (a + c) + i(b + d)$
so,

$$\overline{z+w} = (a + c) - i(b + d) = (a - ib) + (c - id) = \bar{z} + \bar{w}$$

$\square$

(b) $\overline{zw} = \bar{z}\bar{w}$:

Proof. $zw = (ac - bd) + i(ad + bc) \implies \overline{zw} = (ac - bd) - i(ad + bc)$

and so

$$\bar{z}\bar{w} = (a - ib)(c - id) = (ac - bd) - i(ad + bc) = \overline{zw}$$

$\square$

(c) $|zw| = |z||w|$:

Proof.

$$|zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2} = \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2)}$$

$$= \sqrt{a^2(c^2 + d^2) + b^2(c^2 + d^2)}$$

$$= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$$

$$= |z||w|$$

$\square$

(d) $\overline{(\frac{z}{w})} = \frac{\bar{z}}{\bar{w}}$ where $w \neq 0$:

Proof. Note that

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

so

$$\overline{(\frac{z}{w})} = \overline{(\frac{z\bar{w}}{|w|^2})} = \frac{\bar{z}w}{|w|^2} = \frac{\bar{z}}{\bar{w}}$$

$\square$

(e) $|z^n| = |z|^n$:

Proof. Note that this identity holds if $z = 0$.

Assume $z \neq 0$ and $z = re^{i\theta}$ for some $r > 0$ and $\theta \in [-\pi, \pi)$. Then

$$|z^n| = |(re^{i\theta})^n| = |r^n e^{in\theta}| \qquad \text{(de Moivre's formula)}$$

$$= r^n |e^{in\theta}| = r^n = |z|^n$$

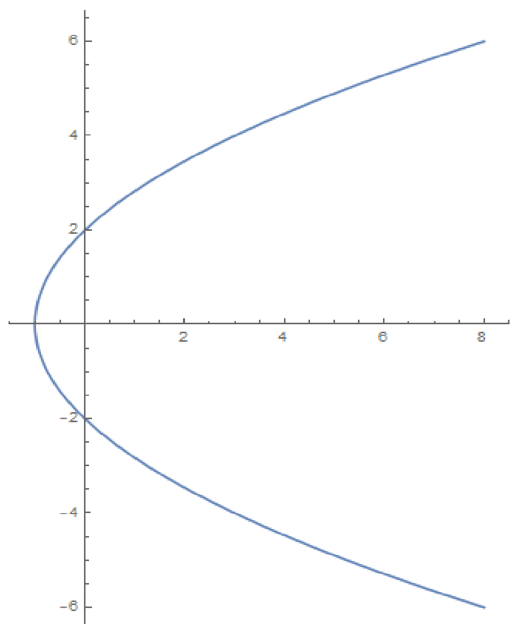
$\square$

Exercise 6. (a) Use Mathematica to plot the image of the line $y = -1$ under the function $f(z) = z^2$:

```
In[1]:= f[z_] := z^2

In[2]:= ParametricPlot[ReIm[f[t + (-1) * I]], {t, -3, 3},
          AspectRatio → Automatic, AxesOrigin → {0, 0}]
```

Out[2]=

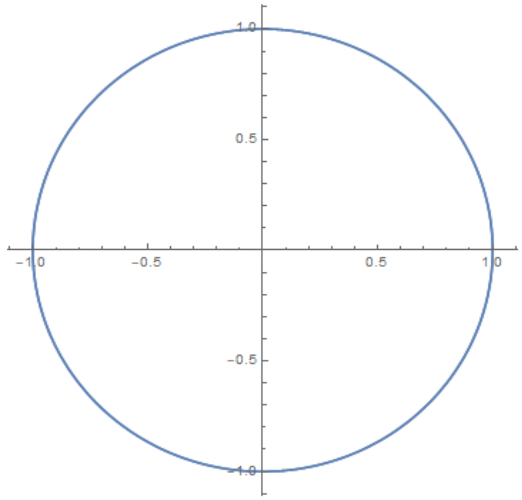


(b) Use Mathematica to plot the image of the unit circle $x^2 + y^2 = 1$ under the function $f(z) = z^2$:

```
In[5]:= f[z_] := z^2

In[6]:= ParametricPlot[ReIm[f[Cos[t] + Sin[t] * I]], {t, 0, 2 * Pi}, AspectRatio → Automatic,
         AxesOrigin → {0, 0}]
```

Out[6]=



(c) Re-do part(a) and (b) for $f(z) = z^3$:

Image of the line $y = -1$ under z^3:

```
In[3]:= f[z_] := z^3

In[4]:= ParametricPlot[ReIm[f[t + (-1) * I]], {t, -3, 3},
         AspectRatio → Automatic, AxesOrigin → {0, 0}]
```

Out[4]=

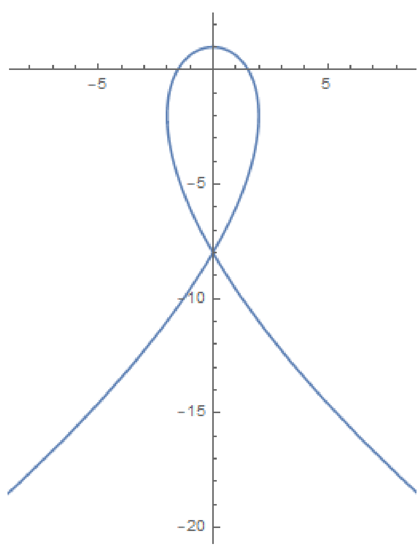
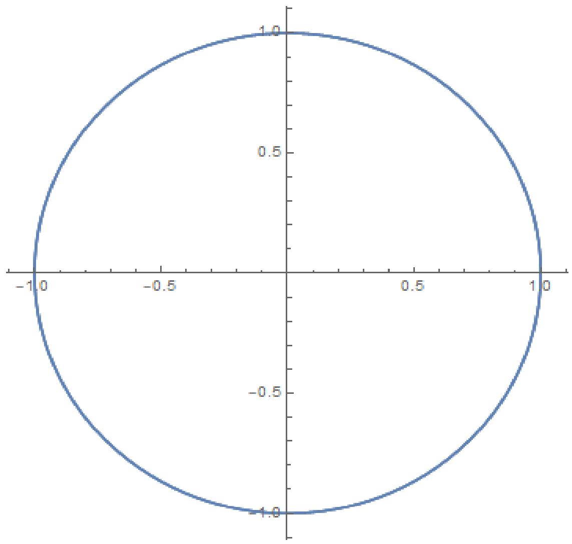


Image of the unit circle $x^2 + y^2 = 1$ under z^3:

In[8]:=
```
ParametricPlot[ReIm[f[Cos[t] + Sin[t] * I]], {t, 0, 2 * Pi},
  AspectRatio → Automatic, AxesOrigin → {0, 0}]
```

Out[8]=



(d) Re-do part (a) and (b) for $f(z) = 1/z$:

Image of the line $y = -1$ under $1/z$:

In[20]:=
```
ParametricPlot[ReIm[f[t - I]], {t, -9, 8}, AspectRatio → Automatic,
  AxesOrigin → {0, 0}]
```

Out[20]=

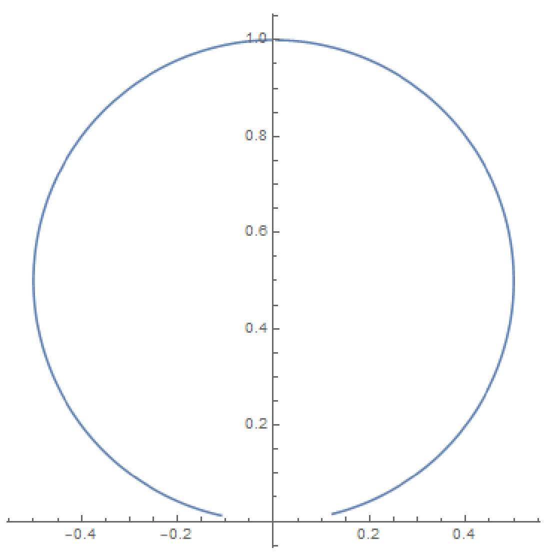


Image of the unit circle $x^2 + y^2 = 1$ under $1/z$:

In[21]:= f[z_] := 1 / z

In[22]:= ParametricPlot[ReIm[f[Cos[t] + Sin[t] * I]], {t, 0, 2 * Pi}, AspectRatio → Automatic,
 AxesOrigin → {0, 0}]

Out[22]=