Homework 3

Due 4/24/2020

1. Write the following complex numbers in either standard or polar form.
   (a) $(1+i\sqrt{3})^{i+1}$
   (b) $(-1)^{\sqrt{2}}$
   (c) $(-1-i)^{1/4}$
   (d) $\log\left(e^{i\frac{19}{4}\pi}\right)$
   (e) $e^{i\sqrt{2}}$

2. Find all complex numbers satisfying the following equation. Write your results in either standard form or polar form.
   (a) $e^z = -1$
   (b) $2^z = 4$
   (c) $\sin z = 1 - 2i$
   (d) $\cos z = 1/2$

3. Decide if each of the following statements is true or false. If it is true, prove it. If it is false, give a counterexample.
   (a) $\mathrm{Log}(z) + \mathrm{Log}(w) = \mathrm{Log}(zw) \quad \forall\, z, w \in \mathbb{C},\ z, w \neq 0.$
   (b) $e^{\mathrm{Log} z} = z \quad \forall\, z \in \mathbb{C},\ z \neq 0.$
   (c) $\sqrt{z^2 - 1} = \sqrt{z-1}\sqrt{z+1} \quad \forall z \in \mathbb{C},\ z \neq \pm 1.$
   Here the principal logarithm is used to compute square root: $\sqrt{w} = w^{1/2} = e^{1/2 \mathrm{Log}\, w}$.

4. Determine all $z \in \mathbb{C}$ such that $z^2 + 1 \in \mathbb{R}_{\leq 0}$. Here $\mathbb{R}_{\leq 0}$ denotes the negative real line.
   Hint: write $z = x + iy$ and then find the conditions on x and y. There is also a geometric method solve this problem.

5. Before doing the following problem, please take a look at the supplemental material called "**Mapping properties of the exponential function (continued)**" posted on Canvas and course website. *Make sure to include the Mathematica codes you use and some brief comments.*
   (a) Draw the image of the rectangle $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-1, 1]$ under $f(z) = \sin z$.
   (b) Draw the images of some horizontal and vertical lines under f. Based on the picture, can you tell what the images of the line $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are?
   (c) f maps the vertical strip $[-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty)$ onto some region Ω. What is Ω? Is the map $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty) \to \Omega$ a one-to-one mapping?
   (d) f maps the open vertical strip $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, \infty)$ onto some region D? What is D? Is the map $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, \infty) \to D$ a one-to-one mapping?