MTH 483/583 Complex Variables – Homework 5

Solution Key

Spring 2020

Exercise 1. Let

$$f(z) = \left(\frac{z+1}{z-1}\right)^i$$

(a) Determine the region of continuity of f. That is, find all $z \in \mathbb{C}$ where f is continuous.

Answer: Rewrite the function $f(z)$

$$\left(\frac{z+1}{z-1}\right)^i = e^{iLog(\frac{z+1}{z-1})}$$

and note that $Log(\frac{z+1}{z-1})$ is discontinuous at those z such that:

$$\frac{z+1}{z-1} \in \mathbb{R}_{\leq 0}$$

Write $z = x + iy$ then

$$\frac{z+1}{z-1} \in \mathbb{R}_{\leq 0} \iff \frac{x^2+y^2-1}{(x-1)^2+y^2} - i\frac{2y}{(x-1)^2+y^2} \in \mathbb{R}_{\leq 0}$$

Thus, $y = 0$ and $x^2 \leq 1$.
Therefore, $f(z)$ is continuous on the region

$$\mathbb{C} \setminus \{z = x + iy \, : \, -1 \leq x \leq 1, y = 0\}$$

$\square$

(b) Pick a point where f is discontinuous. Call it z_0. Use Mathematica to describe how f jumps at z_0.

Answer: Let us pick $z_0 = -\frac{1}{2}$. Then

```
In[93]:= f[z_] := Exp[I*Log[(z + 1) / (z - 1)]]

In[100]:= p[s_] := ParametricPlot[ReIm[-0.5 + t*I], {t, -2, s}, PlotRange → {{-1, 1}, {-2, 2}}]

In[109]:= q[s_] := ParametricPlot[ReIm[f[-0.5 + t*I]], {t, -2, s}, PlotRange → {{0, 15}, {-25, 0.1}}]

In[110]:= Manipulate[{p[s], q[s]}, {s, -1.5, 1.5}]
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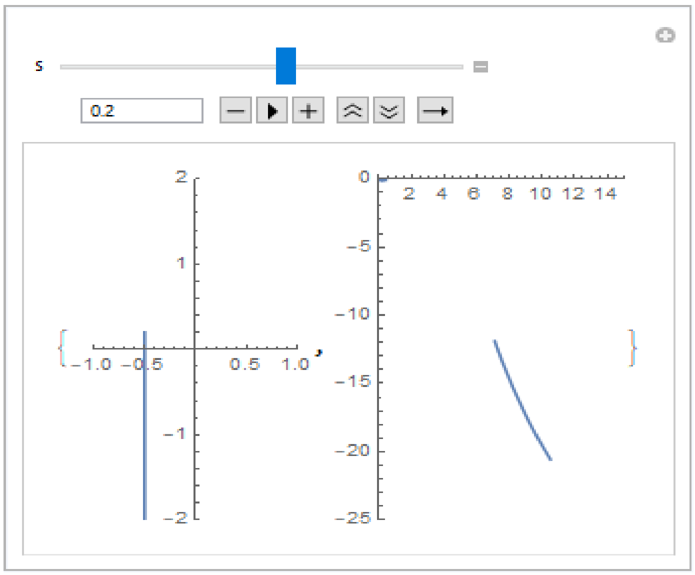
Out[110]=



Explanation: as z crosses the x-axis when moving from south to north, the number $\frac{z+1}{z-1}$ crosses the negative real axis from north to south (see Figure 1). Then $\mathrm{Log}\left(\frac{z+1}{z-1}\right)$ jumps by $-2\pi i$. Then $e^{\mathrm{iLog}\left(\frac{z+1}{z-1}\right)}$ is scaled by $e^{i(-2\pi i)} = e^{2\pi} \approx 535$ times. Before z crosses the x-axis, $\frac{z+1}{z-1} \approx \frac{-1/2+1}{-1/2-1} = -1/3$ but is north of $-1/3$. Thus, $\mathrm{Log}\left(\frac{z+1}{z-1}\right) \approx \ln(1/3) + i\pi$ and $e^{\mathrm{iLog}\left(\frac{z+1}{z-1}\right)} \approx e^{i(\ln(1/3)+i\pi)} \approx 0.0197 + i0.0385$, which is quite a small number. After z crosses the x-axis, this value is suddenly scaled by 535 times. This explains why in the graph we can only observe the big values (the values of f after z crosses the x-axis). The values of f before z crosses the x-axis are too small to see.

```
g[z_] := (z + 1)/(z - 1)
p[s_] := ParametricPlot[ReIm[-1/2 + t*I], {t, -2, s},
  PlotRange -> {{-1, 1}, {-2, 2}}]
q[s_] := ParametricPlot[ReIm[g[-1/2 + t*I]], {t, -2, s},
  PlotRange -> {{-0.5, 1}, {-1, 1}}]
Manipulate[{p[s], q[s]}, {s, -1.9, 1.9}]
```

$\square$

Exercise 2. Determine the region where each of the following function is differentiable. Find the derivative of the function. Determine the region where the function is holomorphic.

(a) $f(z) = \bar{z}^2 z$

Answer: Write $z = x + iy$, then

$$f(z) = (x^3 + xy^2) + i(-x^2y - y^3)$$

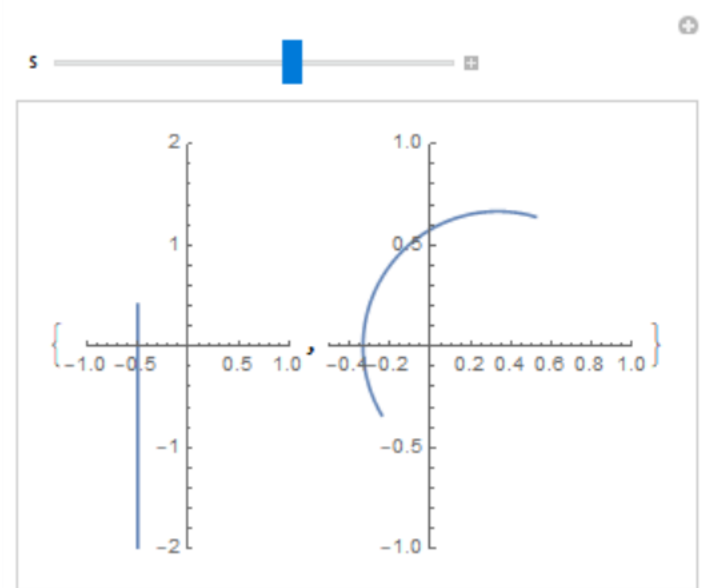


Figure 1:

Let $u(x, y) = x^3 + xy^2$ and $v(x, y) = -x^2y - y^3$
Then Cauchy-Riemann equations

$$u_x = 3x^2 + y^2, \quad u_y = 2xy$$

$$v_x = -2xy, \quad v_y = -x^2 - 3y^2$$

and so

$$u_x = v_y \iff 4x^2 = -4y^2 \iff x = y = 0$$

we also see that $u_y = -v_x$. Thus, since all the partial derivatives exist and are continuous in $\mathbb{C}$, and since the Cauchy-Riemann equations satisfied only at the origin, we conclude that f is differentiable only at the origin, and

$$f'(0) = f_x(0) = 0$$

Finally, f is nowhere holomorphic since f is differentiable only at $z = 0$ and nowhere else.

$\square$

(b) $f(z) = x^2y + x + i(xy^2 - x + y)$, where $z = x + iy$.

Answer: Let $u(x, y) = x^2y + x$, $v(x, y) = xy^2 - x + y$.

Cauchy-Riemann:

$$u_x = 2xy + 1, \quad u_y = x^2$$

$$v_x = y^2 - 1, \quad v_y = 2xy + 1$$

Note that $u_x = v_y$ for all x, y, and

$$u_y = -v_x \Longleftrightarrow x^2 + y^2 = 1$$

Therefore, since the Cauchy-Riemann equations hold for all the points on the unit circle, and all the partial derivatives exist and are continuous in $\mathbb{C}$, $f(z)$ is differentiable on the curve $x^2 + y^2 = 1$.

$$f'(z) = f_x = (2xy + 1) + i(y^2 - 1)$$

Finally, since for each point z_0 on the unit circle and for any open set Ω that contains z_0, $f(z)$ is not differentiable at some points in Ω, so f is nowhere holomorphic.

$\square$

(c) $f(z) = i^z$ (principal logarithm is used)

***Answer**:* There are two ways to do this problem.
Method 1: use composite function.
$f(z) = e^{i\mathrm{Log}z}$ is a composite function.

$$z \xrightarrow{\text{multiply by constant}} z\mathrm{Log}(i) \xrightarrow{\text{exponentiate}} e^{z\mathrm{Log}(i)} = f(z)$$

To be more specific, $f(z) = g(h(z))$ where $g(z) = e^z$ and $h(z) = z\mathrm{Log}(i) = zi\frac{\pi}{2}$. Because g and h are differentiable everywhere, so is f. By the chain rule,

$$f'(z) = g'(h(z))h'(z) = e^{z\mathrm{Log}(i)}\mathrm{Log}(i) = i^z i\frac{\pi}{2} = i^{z+1}\frac{\pi}{2}.$$

Method 2: use Cauchy-Riemann equations.
Let $z = x + iy$, then

$$\begin{aligned} f(z) = i^z &= e^{zLog(i)} = e^{zi\frac{\pi}{2}} \\ &= e^{i\frac{\pi}{2}(x+iy)} = e^{-\frac{\pi}{2}y}\, e^{i\frac{\pi}{2}x} \\ &= e^{-\frac{\pi}{2}y}\cos(\frac{\pi}{2}x) + ie^{-\frac{\pi}{2}y}\sin(\frac{\pi}{2}x) \end{aligned} \tag{1}$$

Let $u(x,y) = e^{-\frac{\pi}{2}y}\cos(\frac{\pi}{2}x)$ and $v(x,y) = e^{-\frac{\pi}{2}y}\sin(\frac{\pi}{2}x)$, then

$$u_x = -\frac{\pi}{2}e^{-\frac{\pi}{2}y}\sin(\frac{\pi}{2}x); \quad u_y = -\frac{\pi}{2}e^{-\frac{\pi}{2}y}\cos(\frac{\pi}{2}x)$$

$$v_x = \frac{\pi}{2}e^{-\frac{\pi}{2}y}\cos(\frac{\pi}{2}x); \quad v_y = -\frac{\pi}{2}e^{-\frac{\pi}{2}y}\sin(\frac{\pi}{2}x)$$

It is easy to see that all the partial derivatives are continuous in $\mathbb{C}$. Moreover,

$$u_x = v_y, \quad u_y = -v_x$$

for every point $z = x + iy$ in $\mathbb{C}$, so the Cauchy-Riemann equations hold everywhere and therefore is entire.

Finally,

$$f'(z) = f_x = -\frac{\pi}{2}e^{-\frac{\pi}{2}y}\sin(\frac{\pi}{2}x) + i\frac{\pi}{2}e^{-\frac{\pi}{2}y}\cos(\frac{\pi}{2}x)$$

$\square$

Exercise 3. Determine the region where the function $f(z) = \tan z$ is differentiable. Then show that $f'(z) = 1 + \tan^2 z$.

***Answer**:* Note that

$$\tan z := \frac{\sin z}{\cos z}$$

and since $\sin z$ and $\cos z$ are entire functions, $\cos z = 0$ when $z = \left(k + \frac{1}{2}\right)\pi$, $k \in \mathbb{Z}$.

Thus $f(z)$ is differentiable in the region

$$\left\{ z \in \mathbb{C} : \ z \neq \left(k + \frac{1}{2}\right)\pi \right\}$$

Using differentiation rules:

$$f'(z) = \left(\frac{\sin z}{\cos z}\right)' = \frac{(\sin z)'\cos z - \sin z(\cos z)'}{\cos^2 z}$$

$$= \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = 1 + \tan^2 z$$

$\square$

Exercise 4. Let $f(z) = \frac{z^2}{z-3i}$ and $z_0 = 0$. Use Mathematica to show that f is not angle-preserving at z_0. What does f do to the angles at z_0 instead?

***Answer**:* We observe that

$$f'(0) = \frac{z^2 - 6iz}{(z-3i)^2} \mid_{z=0} = 0.$$

Because $f'(0) = 0$, we suspect that sf is not angle-preserving at $z = 0$. When z is near 0, $f(z)$ behaves like the function $\frac{z^2}{0-3i}$, which doubles the angle of two curves that meet at the origin. We guess that f also doubles the angle between two curves that meet at the origin. Let us pick two curves to test our guess. Let us choose curve σ to be the half line with $\theta = \pi/4$ and λ to be the half line with $\theta = 0$. That is $\sigma(t) = (1+i)t$ and $\lambda(t) = t$ where $t \geq 0$. The velocity vectors on σ and λ at 0 are

$$\sigma'(0) = 1 + i, \qquad \lambda'(0) = 1.$$

In Mathematica, we can draw these two curves with their velocity vectors at the origin as follows (Figure 2).

```
z0 = 0
sigma[t_] := t*(1 + I)
lambda[t_] := t

u1 = D[sigma[t], t] /. t -> 0
u2 = D[lambda[t], t] /. t -> 0

ParametricPlot[{ReIm[sigma[t]], ReIm[lambda[t]]}, {t, 0, 2},
 PlotRange -> {{-1, 2}, {-1, 2}},
 Epilog -> {Arrow[{ReIm[z0], ReIm[z0] + ReIm[u1]}],
   Arrow[{ReIm[z0], ReIm[z0] + ReIm[u2]}]}]
```

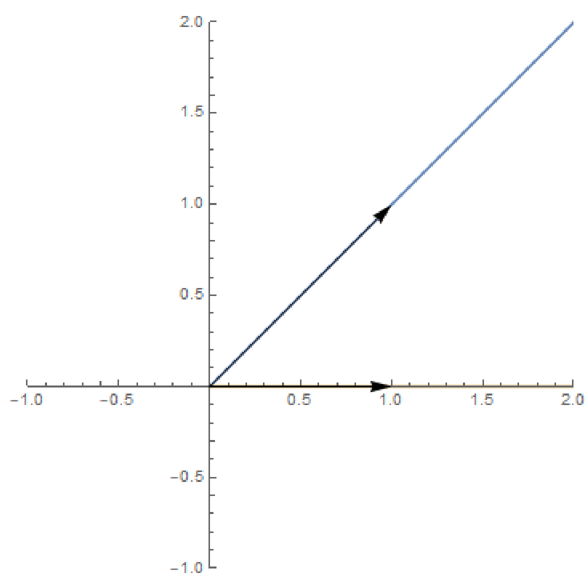


Figure 2:

The image of σ under f is

$$\Sigma(t) = f(\sigma(t)) = f(at) = \frac{a^2t^2}{at - 3i},$$

where $a = 1 + i$. The image of λ under f is

$$\Lambda(t) = f(\lambda(t)) = f(t) = \frac{t^2}{t - 3i}.$$

We can check that the velocity vectors $\Sigma'(0)$ and $\Lambda'(0)$ are both equal to 0. They are not useful tangent vectors on Σ and Λ because we can't use them to compute the angle between these curves. To get nonzero tangent vectors, we have to reparametrize the curves Σ and Λ. The reason why the velocity vectors at $t = 0$ are equal to 0 is because of the factor t^2 in the numerator. We thus reparametrize the curve by $s = t^2$. We get

$$\Sigma_1(s) = \frac{a^2 s}{a\sqrt{s} - 3i}, \qquad \Lambda_1(s) = \frac{s}{\sqrt{s} - 3i}$$

Note that the shape of the curve Σ_1 is the same as Σ. The only difference is the parametrization. Similarly, Λ_1 and Λ differ from each other only by parametrization, not the shape. The velocity vectors on Σ_1 and Λ_1 at $t = 0$ are

$$\Sigma_1'(0) = \left.\frac{a^2(a\sqrt{s} - 3i) - a^2 s \frac{a}{2\sqrt{s}}}{(a\sqrt{s} - 3i)^2}\right|_{s=0} = \frac{a^2}{-3i},$$

$$\Lambda_1'(0) = \frac{1}{-3i}.$$

In Mathematica, we can draw these two curves with their velocity vectors at the origin as follows (Figure 3).

```
f[z_] := z^2/(z - 3*I)
w0 = f[z0]
a = 1 + I
Sigma1[s_] := a^2*s/(a*Sqrt[s] - 3*I)
Lambda1[s_] := s/(Sqrt[s] - 3*I)

v1 = D[Sigma1[s], s] /. s -> 0
v2 = D[Lambda1[s], s] /. s -> 0

ParametricPlot[{ReIm[Sigma1[s]], ReIm[Lambda1[s]]}, {s, 0, 3},
 PlotRange -> {{-1, 1}, {-0.5, 1}},
 Epilog -> {Arrow[{ReIm[w0], ReIm[w0] + ReIm[v1]}],
   Arrow[{ReIm[w0], ReIm[w0] + ReIm[v2]}]}]
```

We see from the graph that the angle between Σ_1 and Λ_1 is $\pi/2$, whereas the angle between σ and λ is $\pi/4$. This experiment confirms our guess that f doubles the angles between two curves that meet at the origin. $\square$

Exercise 5. Consider the function $f(z) = z^3$

(a) Sketch the image of the vertical line (ℓ): $x = 1$ under f. Note that the image curve intersects itself.

```
    f[z_] := z^3
ParametricPlot[ReIm[f[1 + t*I]], {t, -3, 3},
AspectRatio -> Automatic, AxesOrigin -> {0, 0}]
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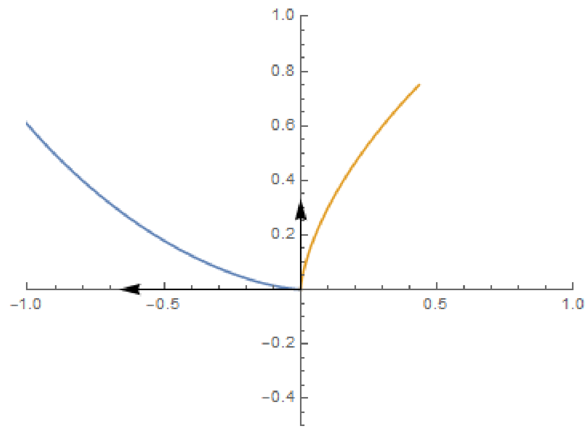
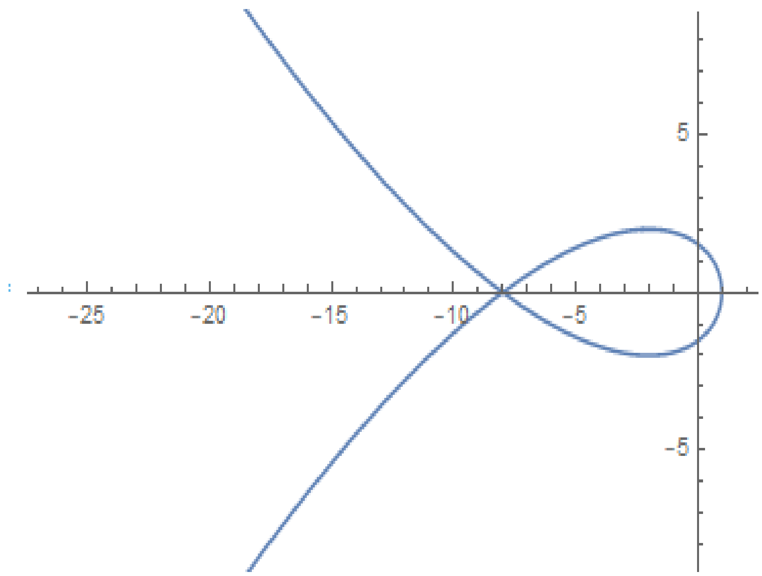


Figure 3:



(b) Find two distinct points z_1 and z_2 on (ℓ) such that $f(z_1) = f(z_2)$.

Answer: Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ where $\theta_1 = \text{Arg} z_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\theta_2 = \text{Arg} z_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Without loss of generality, we may assume that $\theta_1 \geq \theta_2$. Then

$$\left(\frac{z_1}{z_2}\right)^3 = 1 \quad \Longleftrightarrow \left(\frac{r_1}{r_2}\right)^3 e^{i3(\theta_1 - \theta_2)} = 1$$

so

$$\frac{r_1}{r_2} = 1 \quad \text{or} \quad r_1 = r_2$$

and

$$\theta_1 - \theta_2 = \frac{2k\pi}{3}, \quad k \in \mathbb{Z}$$

We can see from Figure 4 that the angle between z_1 and z_2 cannot exceed π. Thus,

$$\theta_1 - \theta_2 = \frac{2\pi}{3}$$

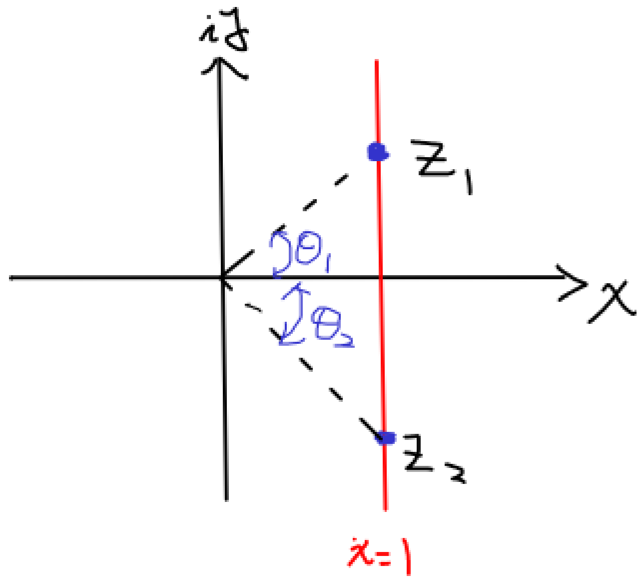


Figure 4:

Furthermore, because $r_1 = r_2$, the triangle with vertices at 0, z_1, z_2 is an isosceles triangle. This implies

$$\theta_1 = -\theta_2$$

Therefore,

$$z_1 = 2e^{i\frac{\pi}{3}} = 1 + i\sqrt{3}, \quad z_2 = 2e^{-i\frac{\pi}{3}} = 1 - i\sqrt{3}$$

$\square$

(c) Find $f'(z_1)$ and $f'(z_2)$.

Answer:

$$f'(z_1) = 3z_1^2 = -6 + 6\sqrt{3}i$$

and

$$f'(z_2) = 3z_2^2 = -6 - 6\sqrt{3}i$$

$\square$

(d) Find the angle at which the image curve intersects itself.

Answer: The angle at the intersection of the curve is

$$\theta = \mathrm{Arg}(f'(z_2)) - \mathrm{Arg}(f'(z_1)) = -\frac{2\pi}{3} - \frac{2\pi}{3} = -\frac{4\pi}{3}$$

which is $2\pi/3$ in modulo 2π.

$\square$