MTH 483/583 Complex Variables – Homework 6

Solution Key

Spring 2020

Exercise 1. Let G be an open subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a holomorphic function. Let $a \in G$. Consider the function

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if} \quad z \neq a, \\ f'(a) & \text{if} \quad z = a. \end{cases}$$

(a) Show that g is a continuous function on G.

***Answer**:* For $z \in G$, $z \neq a$, the function $g(z) = \frac{f(z)-f(a)}{z-a}$ is a multiplication of two continuous functions, namely, $f(z) - f(a)$ and $\frac{1}{z-a}$. Thus g is also continuous at any $z \neq a$. Let us show the continuity of g at $z = a$. We have

$$\lim_{z \to a} g(z) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = f'(a) = g(a).$$

Therefore, g is continuous at $z = a$. In conclusion, g is continuous on G. □

(b) Let us write f in standard form as $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. Write g in standard form.

***Answer**:* We may assume $a = 0$ for simplicity.

Let us simply write $u(x, y)$ and $u(0, 0)$ as u and u_0 respectively. Similarly for v and v_0. For $z \neq 0$, we have

$$g(z) = \frac{f(z) - f(0)}{z} = \frac{(u + iv) - (u_0 + iv_0)}{x + iy} = \frac{(u - u_0) + i(v - v_0)}{x + iy}$$

$$= \frac{x(u - u_0) + y(v - v_0)}{x^2 + y^2} + i\,\frac{x(v - v_0) - y(u - u_0)}{x^2 + y^2}$$

For $z = 0$,

$$g(a) = g(0) = f'(0) = (\partial_x u)(0, 0) + i\,(\partial_x v)(0, 0)$$

Therefore,

$$g(z) = \begin{cases} \dfrac{x(u-u_0)+y(v-v_0)}{x^2+y^2} + i\,\dfrac{x(v-v_0)-y(u-u_0)}{x^2+y^2} & z \neq 0 \\ (\partial_x u)(0,0) + i\,(\partial_x v)(0,0) & z = 0. \end{cases}$$

□

(c) Show that g is holomorphic on G.

Answer: Because $f(z)-f(a)$ is differentiable on G and $\frac{1}{z-a}$ is differentiable on $G\backslash\{0\}$, the product $\frac{f(z)-f(a)}{z-a}$ is differentiable on $G\backslash\{0\}$. Therefore, g is differentiable at every $z_0 \in G$, $z_0 \neq 0$.

We now show that g is differentiable at a. Let us assume $f(a) = a = 0$ for simplicity. We will show that the Cauchy–Riemann equations are satisfied at $(0,0)$. Based on the formula we obtained in part (b), $g(z) = U(x,y) + iV(x,y)$ where

$$U(x,y) = \begin{cases} \dfrac{xu(x,y)+yv(x,y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ (\partial_x u)(0,0), & (x,y) = (0,0). \end{cases}$$

$$V(x,y) = \begin{cases} \dfrac{xv(x,y)-yu(x,y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ (\partial_x v)(0,0), & (x,y) = (0,0). \end{cases}$$

Then

$$\begin{aligned} \partial_x U(0,0) &= \lim_{h\to 0} \frac{U(h,0)-U(0,0)}{h} = \lim_{h\to 0} \frac{\frac{u(h,0)}{h} - \partial_x u(0,0)}{h} \\ &= \lim_{h\to 0} \frac{u(h,0) - h\partial_x u(0,0)}{h^2} \\ &= \lim_{h\to 0} \frac{\partial_x u(h,0) - \partial_x u(0,0)}{2h} \quad \text{(L'Hopital Rule)} \\ &= \frac{1}{2}\partial_{xx} u(0,0) \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} \partial_y V(0,0) &= \lim_{h\to 0} \frac{V(0,h)-V(0,0)}{h} = \lim_{h\to 0} \frac{-\frac{u(0,h)}{h} - \partial_x v(0,0)}{h} \\ &= \lim_{h\to 0} \frac{-u(0,h) + h\partial_y u(0,0)}{h^2} \\ &= \lim_{h\to 0} \frac{-\partial_y u(0,h) + \partial_y u(0,0)}{2h} \quad \text{(L'Hopital Rule)} \\ &= -\frac{1}{2}\partial_{yy} u(0,0) \end{aligned} \tag{2}$$

According to Problem 4, $\partial_{xx}u = -\partial_{yy}u$. Thus, $\partial_x U(0,0) = \partial_y V(0,0)$. Similarly,

$$\begin{aligned}
\partial_y U(0,0) &= \lim_{h\to 0} \frac{U(0,h) - U(0,0)}{h} = \lim_{h\to 0} \frac{\frac{v(0,h)}{h} - \partial_x u(0,0)}{h} \\
&= \lim_{h\to 0} \frac{v(0,h) - h\partial_y v(0,0)}{h^2}, \quad \text{(Cauchy-Riemann eqn's for u and v)} \\
&= \lim_{h\to 0} \frac{\partial_y v(0,h) - \partial_y v(0,0)}{2h} \quad \text{(L'Hopital Rule)} \\
&= \frac{1}{2}\partial_{yy} v(0,0)
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
\partial_x V(0,0) &= \lim_{h\to 0} \frac{V(h,0) - V(0,0)}{h} = \lim_{h\to 0} \frac{\frac{v(h,0)}{h} - \partial_x v(0,0)}{h} \\
&= \lim_{h\to 0} \frac{v(h,0) - h\partial_x v(0,0)}{h^2} \\
&= \lim_{h\to 0} \frac{\partial_x v(h,0) - \partial_x v(0,0)}{2h} \quad \text{(L'Hopital Rule)} \\
&= \frac{1}{2}\partial_{xx} v(0,0)
\end{aligned} \tag{4}$$

According to Problem 4, $\partial_{xx}v = -\partial_{yy}v$. Thus, $\partial_y U(0,0) = -\partial_x V(0,0)$.

So g is differentiable at 0. We have showed that g is differentiable everywhere in G. We now explain why g is holomorphic on G.

For each $z_0 \in G$, there is an open disk $D_r(z_0)$ centered at z_0 with radius r that lies entirely in G. This is because G is an open set. We know that g is differentiable everywhere in this disk. Thus, g is holomorphic at z_0. Because z_0 is arbitrary in G, we conclude that g is holomorphic on G.

□

Exercise 2. Let G be an open connected subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a *holomorphic* map on G. Suppose $f'(z) = 0$ for all $z \in G$. We want to show that f is a constant function. Follow the steps:

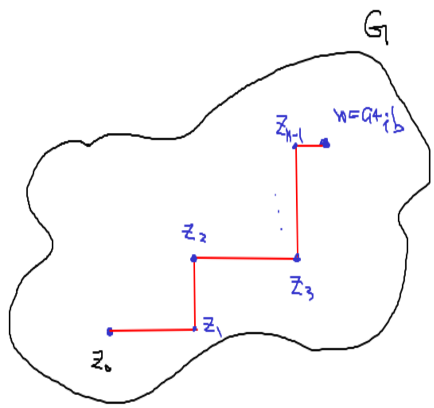
(a) Write f in standard form $f(z) = u(x,y) + iv(x,y)$. Show that $u_x = u_y = v_x = v_y = 0$ in G.

Answer: We know that $f'(z) = u_x + iv_x$. Therefore, $u_x = v_x = 0$ everywhere in G. By Cauchy–Riemann equations, $u_y = -v_x = 0$ and $v_y = u_x = 0$.

□

(b) Show that u and v are constant functions.

Answer: Fix $z_0 = x_0 + iy_0 \in G$. Let $w = a + ib$ be an arbitrary point in G. Since G is open and connected, there is a rectilinear curve lies entirely in G that starts at z_0 and ends at w. This rectilinear curve consists of horizontal and vertical line segments, let us denote the joint points by $z_k = x_k + iy_k$, $k = 0, 1, ..., n$, where $z_n = a + ib$.(See the graph below)



In other words, each line segment $[z_k, z_{k+1}]$, $k = 0, 1, ...n-1$, is either a horizontal or vertical line.

If the segment that connect z_k to z_{k+1} is horizontal then $u(x_{k+1}, y_{k+1}) = u(x_k, y_k)$ because $u_x = 0$. If the segment that connect z_k to z_{k+1} is vertical then $u(x_{k+1}, y_{k+1}) = u(x_k, y_k)$ because $u_y = 0$. Hence,

$$u(x_{k+1}, y_{k+1}) = u(x_k, y_k)$$

Since this holds for all $k \in \{0, 1, 2, ..., n\}$, we have

$$u(x_0, y_0) = u(x_1, y_1) = \cdots = u(x_n, y_n) = u(a, b)$$

Similarly, $v(x_0, y_0) = v(a, b)$

Finally, since this holds for arbitrary $w = a + ib \in G$, we conclude that u and v are constant functions on G. $\square$

(c) Is f necessarily a constant function if the condition "G is a connected subset" is dropped?

Answer: Let

$$A := \{z = x + iy : x < -1\}$$

and

$$B := \{z = x + iy : x > 1\}$$

Define $G := A \cup B$. See Figure 2.

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RegionPlot[x < -1 || x > 1, {x, -3, 3}, {y, -2, 2},
 AspectRatio -> Automatic]
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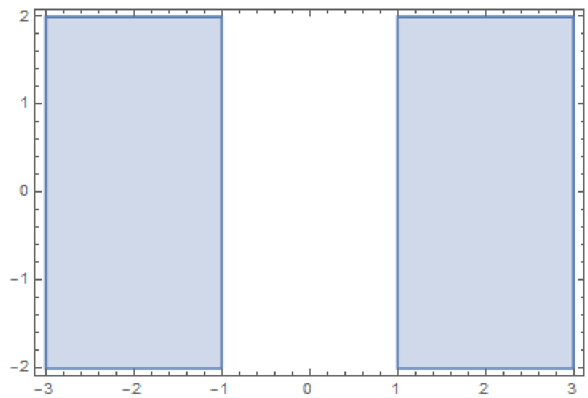

Figure 1:

A point in set A cannot be connected to a point in set B by a path that lies in G. Therefore, G is not connected.

Consider the function

$$f(z) = \begin{cases} 0 & \text{if} \quad z \in A \\ 1 & \text{if} \quad z \in B \end{cases}$$

We see that $f'(z) = 0$. However, f is not constant in G (although it is constant in A and B). $\square$

Exercise 3. Consider $f(z) = \frac{1}{z}$. We know that $F(z) = \text{Log } z$ is an antiderivative of f. To be more precisely, F is an antiderivative of f in the region $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. In this problem, we will see that f has many other antiderivatives (differing f by a non-constant) in other regions.

(a) Show that any antiderivative of f in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ must be $F(z) + c$ where c is a complex constant.

***Answer**:* Let $G(z)$ be any antiderivative of f in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Note that $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is open and connected and G is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Then

$$(G - F)'(z) = G'(z) - F'(z) = \frac{1}{z} - \frac{1}{z} = 0$$

Since $G(z) - F(z)$ is also holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, Problem 2 tells us that $G(z) - F(z)$ is constant, say, c. Thus, $G(z) - F(z) = c$ or $G(z) = F(z) + c$.

$\square$

(b) For $\theta \in (-\pi, \pi]$, denote $G(z) = \text{Log } (e^{i\theta} z)$. Describe the region of continuity of G. Show that $G' = f$ in this region. Is the difference $G - F$ a constant function?

***Answer**:* $\text{Log}(e^{i\theta} z)$ is discontinuous at those z such that

$$e^{i\theta} z \in \mathbb{R}_{\leq 0}$$

write $z = re^{i\beta}$, then

$$e^{i\theta}z = re^{i(\theta+\beta)} \in \mathbb{R}_{\leq 0} \quad \Longleftrightarrow \quad \theta + \beta = \pi \pmod{2\pi}$$

Thus, G is continuous everywhere except on the ray $\text{Arg}z = \pi - \theta$ (the green line in Figure 2):

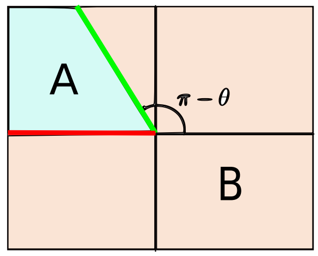

Figure 2:

Using differentiation rule

$$G'(z) = \left(\text{Log}(e^{i\theta}z)\right)' = \frac{1}{e^{i\theta}z}e^{i\theta} = \frac{1}{z}$$

which shows that $G' = f$ in the region $\mathbb{C} \setminus \left\{z = re^{i\beta} : \beta = \pi - \theta\right\}$.

$G - F$ is a holomorphic function in the region $\Omega = A \cup B$ as shown in Figure 2. It is not a constant function on Ω unless $\theta = 0$. To see this, let us take $\theta = \pi$ for example.

For $z_1 = e^{-i\frac{\pi}{2}}$, we have

$$\text{Log}(e^{i\theta}z_1) = \text{Log}(e^{i\frac{\pi}{2}}) = i\frac{\pi}{2}$$

so $G(z_1) - F(z_1) = i\frac{\pi}{2} - i\frac{-\pi}{2} = i\pi$.

For $z_2 = e^{-i\frac{3\pi}{2}}$, we have

$$\text{Log}(e^{i\theta}z_2) = \text{Log}(e^{-i\frac{\pi}{2}}) = -i\frac{\pi}{2}$$

so $G(z_2) - F(z_2) = -i\frac{\pi}{2} - i\frac{\pi}{2} = -i\pi$.

Thus, $G - F$ is not a constant function on Ω. However, it is a constant function on A and on B. Note that Ω is not a connected set because it is impossible to connect a point in A to a point in B by any continuous path. This is another example (in addition to the example given in Problem 2, Part (c)) to show that a function whose derivative is equal to zero on a disconnected set may not be a constant function.

$\square$

(c) Show that f has no antiderivatives in the region $\mathbb{C} \setminus \{0\}$.

***Answer**:* Suppose by contradiction that there is an antiderivative of f in the region $\mathbb{C} \setminus \{0\}$. Let us call it $H(z)$. We have $H'(z) = f(z)$ for all $z \in \mathbb{C} \setminus \{0\}$.

In particular, $H(z)$ is an antiderivative of $f(z)$ in the region $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. According to Part (a), $H(z) = F(z) + c$ in the region $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Because F is discontinuous on the negative real line (jumping by $2\pi i$ across the negative real line), so must be H. On the other hand, H is holomorphic everywhere on $\mathbb{R}_{<0}$, so it must be continuous everywhere on $\mathbb{R}_{<0}$. This is a contradiction.

□

A function $u(x, y)$ is said to be **harmonic** in a region G if the *Laplacian* $\Delta u = u_{xx} + u_{yy}$ is equal to zero for all $(x, y) \in G$.

Exercise 4. Let $f(z) : G \to \mathbb{C}$ be a holomorphic function on G. Show that the real part and imaginary part of f are harmonic functions.

Answer: Write $f(z) = u(x, y) + iv(x, y)$.
Since f is differentiable in G, the Cauchy-Riemann equations holds:

$$u_x = v_y; \quad u_y = -v_x$$

Thus,

$$u_{xx} + u_{yy} = v_{xy} + (-v_{yx}) = 0$$

and

$$v_{xx} + v_{yy} = -u_{xy} + u_{yx} = 0$$

□

Exercise 5. Find an entire function f such that $f(0) = 1 - 2i$ and the real part of f is

$$u(x, y) = e^{-y} \cos(x) - y$$

Answer: Let $v(x, y)$ be the imaginary part of $f(z)$. Since f is entire, it must satisfy the Cauchy-Riemann equations. Thus,

$$v_x = -u_y = e^{-y} \cos(x) + 1 \tag{5}$$

and

$$v_y = u_x = -e^{-y} \sin(x) \tag{6}$$

Integrating (5) with respect to x, we get

$$v(x, y) = e^{-y} \sin(x) + x + C(y).$$

Differentiate both sides with respect to y:

$$v_y = -e^{-y} \sin(x) + C'(y).$$

Comparing this equation with (6), we get $C'(y) = 0$. Thus, $C(y) = c$ for some constant c. We obtain

$$v(x, y) = e^{-y} \sin(x) + x + c$$

To find c, we use the fact that $f(0) = 1 - 2i$. This equation implies $c = -2$. Therefore,

$$v(x, y) = e^{-y} \sin(x) + x - 2$$

So,

$$f(z) = \left(e^{-y} \cos(x) - y\right) + i \left(e^{-y} \sin(x) + x - 2\right).$$

If one wishes to obtain a neat formula in terms of z, one can proceed as follows:

$$\begin{aligned} f(z) &= e^{-y}(\cos x + i \sin x) + (-y + ix) - 2i \\ &= e^{-y+ix} + (-y + ix) - 2i \\ &= e^{i(x+iy)} + i(x + iy) - 2i \\ &= e^{iz} + iz - 2i. \end{aligned}$$

The above procedure is simply cosmetic! $\square$