Homework 6

Due 5/22/2020

A subset $G \subset \mathbb{C}$ is said to be ***open*** *if for every $z \in G$ there is a disk $D_r(z)$ (centered at z with radius $r > 0$) that lies entirely in G.*

1. Let G be an open subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a holomorphic function. Let $a \in G$. Consider the function

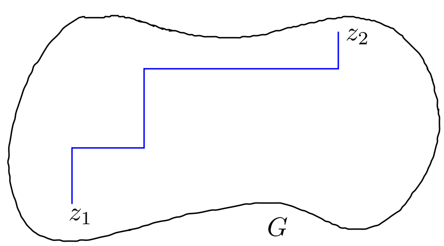
$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } \quad z \neq a, \\ f'(a) & \text{if } \quad z = a. \end{cases}$$

   (a) Show that g is a continuous function on G.

   (b) Let us write f in standard form as $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$. Write g in standard form. *(For the sake of simplicity, you can assume $f(a) = a = 0$.)*

   (c) Show that g is holomorphic on G.
   Hint: explain why g is differentiable on $D\backslash\{a\}$. Then use Cauchy–Riemann equations to explain why g is differentiable at a. (For the sake of simplicity, you can assume $f(a) = a = 0$.)

An open subset $G \subset \mathbb{C}$ is said to be ***connected*** *if for any two points $z_1, z_2 \in G$ there is a rectilinear curve inside G that starts at z_1 and ends at z_2. A rectilinear curve is a curve that consists of horizontal and vertical line segments. See the figure.*



2. Let G be an open and connected subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a holomorphic map on G. Suppose $f'(z) = 0$ for all $z \in G$. We want to show that f is a constant function. Follow the steps:

   (a) Write f in standard form $f(z) = u(x,y) + iv(x,y)$. Show that $u_x = u_y = v_x = v_y = 0$ in G.

   (b) Show that u and v are constant functions.
   Hint: Fix a point $z_0 = x_0 + iy_0$ in G. Let $w = a + ib$ be an arbitrary point in G. Explain why $u(a,b) = u(x_0, y_0)$. Explain likewise for v.

   (c) Is f is necessarily a constant function if the condition "G is a connected subset" is removed?

3. Consider $f(z) = \frac{1}{z}$. We know that $F(z) = \operatorname{Log} z$ is an antiderivative of f. To be more precisely, F is an antiderivative of f in the region $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$. In this problem, we will see that f has many other antiderivatives (differing f by a non-constant) in other regions.

   (a) Show that any antiderivative of f in $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ must be $F(z) + c$ where c is a complex constant.
   Hint: use Problem 2.

(b) For $\theta \in (-\pi, \pi]$, denote $G(z) = \mathrm{Log}(e^{i\theta} z)$. Describe the region of continuity of G. Show that $G' = f$ in this region. Is the difference $G - F$ a constant function?

(c) Show that f has no antiderivatives in the region $\mathbb{C} \backslash \{0\}$.
Hint: suppose by contradiction that it has an antiderivative H in this region. Use Part (a) to show that $R(z) = H(z) - F(z)$ is constant on $\mathbb{C} \backslash \{0\}$.

A function $u(x, y)$ is said to be ***harmonic*** *in a region G if the Laplacian $\Delta u = u_{xx} + u_{yy}$ is equal to zero for all $(x, y) \in G$.*

4. Let $f : G \to \mathbb{C}$ be a holomorphic function on G. Show that the real part and imaginary part of f are harmonic functions.

5. Find an entire function f (i.e. f is holomorphic on $\mathbb{C}$) such that $f(0) = 1 - 2i$ and the real part of f is

$$u(x, y) = e^{-y} \cos(x) - y.$$