MTH 483/583 Complex Variables – Homework 7

Solution Key

Spring 2020

Exercise 1. Let $f(z) = u(x,y) + iv(x,y)$ be a holomorphic function in region $D \subset \mathbb{C}$. Show that the vector field $P(x,y) = (u(x,y), -v(x,y))$ is incompressible and irrotational. In other words, show that $\text{div}P = 0$ and $\text{curl}P = 0$.

Answer: Since f is holomorphic in D, the Cauchy-Riemann equations hold

$$u_x = v_y, \quad u_y = -v_x \quad \text{in } D$$

Thus,

$$\text{div}\, P = \partial_x u(x,y) + \partial_y(-v(x,y)) = u_x - v_y = 0, \quad (x,y) \in D$$

and

$$\text{curl}\, P = \partial_x(-v(x,y)) - \partial_y(u(x,y)) = -v_x - u_y = 0, \quad (x,y) \in D$$

$\square$

Exercise 2. Let $\Phi(z) = \phi(x,y) + i\psi(x,y)$ be an antiderivative of function $f(z) = u(x,y) + iv(x,y)$ in the region $D \in \mathbb{C}$. Show that at each point $(x,y) \in D$ the vector $P(x,y) = (u(x,y), -v(x,y))$ is tangent to the level set of ψ that passes through (x,y)

Answer: Fix $(x_0, y_0) \in D$. The level set of ψ that passes through (x_0, y_0) is the curve $\ell = \{(x,y) : \psi(x,y) = C\}$, where $C = \psi(x_0, y_0)$. We have

$$\Phi'(z) = \phi_x + i\psi_x$$
$$f(z) = u + iv$$

Because Φ is an antiderivative of f, we have $\Phi'(z) = f(z)$. In other words,

$$\phi_x = u, \quad \psi_x = v$$

Because Φ is holomorphic, the Cauchy-Riemann equations hold for ϕ and ψ. In particular,

$$u = \phi_x = \psi_y$$

Then

$$P(x_0, y_0) = (u(x_0, y_0), -v(x_0, y_0)) = (\psi_y(x_0, y_0), -\psi_x(x_0, y_0))$$

We see that $P(x_0, y_0)\cdot\nabla\psi(x_0, y_0) = \psi_y\psi_x - \psi_x\psi_y = 0$. This implies $P(x_0, y_0)$ is perpendicular to the gradient vector $\nabla\psi(x_0, y_0)$. On the other hand, the level set ℓ of ψ is also perpendicular to the gradient vector of $\nabla\psi$ at (x_0, y_0). Therefore, $P(x_0, y_0)$ is tangent to the level set of ψ at (x_0, y_0). □

Exercise 3. Determine an antiderivative of the following function. Then identify the region where that antiderivative is holomorphic.

(a) $f(z) = \frac{z}{z^2+1}$

Answer: We will find an antiderivative of f by the substitution rule as we would normally do with the real-variable function $\frac{x}{x^2+1}$. Put $u = z^2 + 1$. Then $du = 2zdz$.

$$\int \frac{z}{z^2+1}dz = \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\operatorname{Log} u = \frac{1}{2}\operatorname{Log}(z^2+1).$$

Thus, $F(z) = \frac{1}{2}\operatorname{Log}(z^2+1)$ is an antiderivative of f. The region where F is discontinuous is for those z such that

$$z^2 + 1 \in \mathbb{R}_{\leq 0}$$

Write $z = x + iy$ then

$$z^2 + 1 \in \mathbb{R}_{\leq 0} \quad\Longleftrightarrow\quad x = 0,\, y \geq 1 \,\text{or}\, y \leq -1$$

Thus, F is holomorphic in the region

$$\Omega = \mathbb{C} \setminus \{z = iy : |y| \geq 1\}.$$

You can sketch a picture of Ω on the complex plane to see what it looks like.

Important notes: Although *all* antiderivatives of the real-variable function $\frac{x}{x^2+1}$ are of the form $\frac{1}{2}\ln(x^2+1) + C$, not all antiderivatives of the complex-variable function $\frac{z}{z^2+1}$ are of the form $\frac{1}{2}\operatorname{Log}(z^2+1) + C$. Only the antiderivatives of f in the region Ω have to be of the form $\frac{1}{2}\operatorname{Log}(z^2+1) + C$ (see Problem 3 of Homework 6). The reason is that antiderivative of $\frac{1}{u}$ is $\operatorname{Log}(au)$ where a is any complex constant. This is not the same as $\operatorname{Log}(u) + \text{const}$. The domain of continuity and differentiability of $\operatorname{Log}(au)$ is different from that of $\operatorname{Log}(u)$. □

(b) $f(z) = z\sin(z^2+1)$

Answer: We will find an antiderivative of f by the substitution rule as we would normally do with the real-variable function $x\sin(x^2+1)$. Put $u = z^2 + 1$. Then $du = 2zdz$.

$$\int z\sin(z^2+1)dz = \int (\sin u)\left(\frac{1}{2}du\right) = \frac{1}{2}\cos u = \frac{1}{2}\cos(z^2+1).$$

Thus, $F(z) = \frac{1}{2}\cos(z^2+1)$ is an antiderivative of f. Since F is the composition of two entire functions, namely, the cosine function $\cos(z)$, and the polynomial z^2+1. Hence, F is holomorphic on the entire complex plane.

$\square$

(c) $f(z) = z^2 \text{Log} z$

Answer: We will find an antiderivative of f by integration by part as we would normally do with the real-variable function $x^2 \ln x$.

$$\begin{aligned} u &= \operatorname{Log} z, & dv &= z^2 dz \\ du &= \frac{1}{z} dz, & v &= \frac{z^3}{3} \end{aligned}$$

We have

$$\int z^2 \operatorname{Log} z dz = \int u dv = uv - \int v du = \frac{1}{3} z^3 \operatorname{Log} z - \int \frac{z^2}{3} dz = \frac{1}{3} z^3 \operatorname{Log} z - \frac{z^3}{9}.$$

Thus, $F(z) = \frac{1}{3}z^3\text{Log}(z) - \frac{1}{9}z^3$ is an antiderivative of f. Note that the first term of f is the multiplication of z^3 and $\text{Log}z$ and the second term is an entire function $-\frac{1}{9}z^3$. Thus F is holomorphic in the region

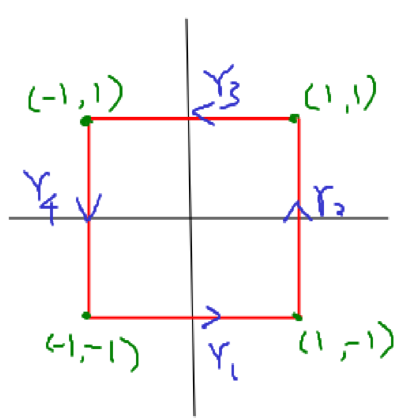
$$\mathbb{C} \setminus \mathbb{R}_{\leq 0}$$

$\square$

Exercise 4. Evaluate the complex integrals $\int_\gamma f(z)$ where f and γ are given as follows. Clearly mention the method/theorem you use.

(a) $f(z) = \bar{z}$ and γ is the square with vertices at $(-1,-1), (1,-1), (1,1), (-1,1)$ positively oriented.

Answer: We will use the definition of line integral (because f has no antiderivatives). Let us divide γ into 4 pieces: $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ (see the graph below)



where

$$\begin{aligned}
\gamma_1(t) &= t - i, \quad t \in [-1,1] \\
\gamma_2(t) &= 1 + it, \quad t \in [-1,1] \\
\gamma_3(t) &= -t + i, \quad t \in [-1,1] \\
\gamma_4(t) &= -1 - it, \quad t \in [-1,1]
\end{aligned}$$

Then

$$\int_\gamma \bar{z}\, dz = \int_{\gamma_1} \bar{z}\, dz + \int_{\gamma_2} \bar{z}\, dz + \int_{\gamma_3} \bar{z}\, dz + \int_{\gamma_4} \bar{z}\, dz$$

$$= \int_{-1}^{1} \overline{\gamma_1(t)}\gamma_1'(t)dt + \int_{-1}^{1} \overline{\gamma_2(t)}\gamma_2'(t)dt + \int_{-1}^{1} \overline{\gamma_3(t)}\gamma_3'(t)dt + \int_{-1}^{1} \overline{\gamma_4(t)}\gamma_4'(t)dt$$

$$= \int_{-1}^{1} \overline{t-i}\, dt + \int_{-1}^{1} \overline{1+it}\,(i)\, dt + \int_{-1}^{1} \overline{-t+i}(-1)\, dt + \int_{-1}^{1} \overline{-1-it}(-i)\, dt$$

$$= \int_{-1}^{1} (t+i)dt + \int_{-1}^{1} (1-it)idt + \int_{-1}^{1} (-t-i)(-1)dt + \int_{-1}^{1} (-1+it)(-i)dt$$

$$= \boxed{8i}$$

□

(b) $f(z) = |z|^2$ and γ is the ellipse $x^2 + \frac{y^2}{4} = 1$ negatively oriented.

***Answer*:** One can plot the ellipse with the command **ContourPlot** in Mathematica (Figure 1). This command plots the solution set of an equation (or system of equations).

```
ContourPlot[{x^2 + y^2/4 == 1}, {x, -2, 2}, {y, -2, 2}, Axes -> True]
```

The ellipse has parametrization $x(t) = \cos t$, $y(t) = 2\sin t$ where $0 \le t \le 2\pi$. Unfortunately, this parametrization gives a positive orientation on the ellipse. One has to reverse the orientation. To do so, we simply replace t with $0 + 2\pi - t = 2\pi - t$. Now we have $x(t) = \cos(2\pi - t) = \cos t$ and $y(t) = 2\sin(2\pi - t) = -2\sin t$. Thus,

$$\gamma(t) = \cos t - 2i \sin t, \quad t \in [0, 2\pi].$$

Thus,

$$\int_\gamma |z|^2 dz = \int_0^{2\pi} |\gamma(t)|^2 \gamma'(t)dt = \int_0^{2\pi} \left(\cos^2 t + 4\sin^2 t\right)(-\sin t - 2i\cos t)dt.$$

One can split the integral into the real part and imaginary part:

$$\int_\gamma |z|^2 dz = -\int_0^{2\pi} \left(\cos^2 t + 4\sin^2 t\right) \sin t dt - 2i \int_0^{2\pi} \left(\cos^2 t + 4\sin^2 t\right) \cos t dt$$

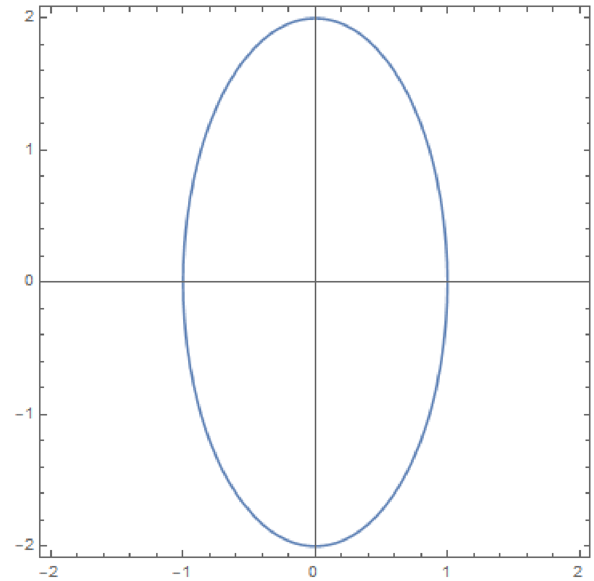


Figure 1:

One can use the fact that $\sin^2 t = 1 - \cos^2 t$ and $\cos^2 t = 1 - \sin^2 t$ to compute each integral. Each is equal to zero. Thus,

$$\int_\gamma f(z)dz = \boxed{0}$$

$\square$

(c) $f(z) = \frac{z^2+1}{z^3+3z+1}$ and γ is the part of parabola $y = x^2$ from $x = 0$ to $x = 2$.

Answer: We use the substitution $w = z^3 + 3z + 1$. Note that $dw = 3(z^2 + 1)dz$. Thus,

$$\int_\gamma \frac{z^2+1}{z^3+3z+1} dz = \frac{1}{3}\int_\eta \frac{1}{w} dw \tag{1}$$

where η is the image of γ under the function $g(z) = z^3 + 3z + 1$. Knowing that γ has parametrization

$$\gamma(t) = x(t) + iy(t) = t + it^2, \quad t \in [0, 2]$$

we can draw η by Mathematica as follows (Figure 2).

```
g[z_] := z^3 + 3*z + 1
ParametricPlot[ReIm[g[t + I*t^2]], {t, 0, 2}]
```

In other words, $\eta(t) = g(\gamma(t)) = \gamma(t)^3 + 3\gamma(t) + 1$. The endpoints of η are

$$\eta(0) = \gamma(0)^3 + 3\gamma(0) + 1 = 1$$

$$\eta(2) = \gamma(2)^3 + 3\gamma(2) + 1 = -81 - 4i$$

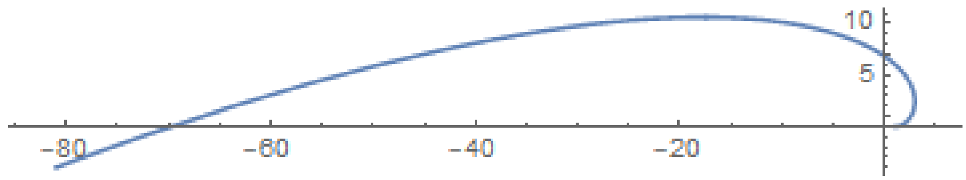


Figure 2:

We see that η intersects the negative real line. This implies that η doesn't lie entirely in the region where Log$w$ is holomorphic. One needs to choose a different antiderivative of $\frac{1}{w}$ so that the η lies entirely in the region of holomorphicity of that function. Let us choose the following antiderivative of $\frac{1}{w}$:

$$G(w) = \mathrm{Log}\left(e^{-i\frac{\pi}{2}}w\right) = \mathrm{Log}(-iw)$$

The region where $G(w)$ is an antiderivative of $1/w$ is the whole complex plane except for the red line in Figure 3. This region contains the entire η. Thus, one can use FTC at (1):

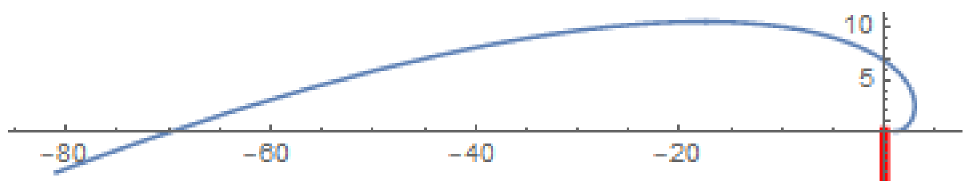


Figure 3:

$$\int_\eta \frac{1}{w}dw = G(\eta(2)) - G(\eta(0)) = G(-81-4i) - G(1) = \mathrm{Log}(-4+81i) - \mathrm{Log}(-i) \approx 4.3957 + 3.1909\mathrm{i}$$

Therefore,

$$\int\limits_\gamma \frac{z^2+1}{z^3+3z+1}dz = \frac{1}{3}\int\limits_\eta \frac{1}{w}dw \approx 1.4652 + 1.0637i$$

$\square$

(d) $f(z) = z^3 e^z$ where γ is the part of the hyperbola $y = \frac{1}{x}$ from $x = 2$ to $x = 1$.

***Answer**:* By doing integration by part three times (in the manner of Problem 3c), we see that $F(z) = (z^3 - 3z^2 + 6z - 6)e^z$ is an antiderivative of f in $\mathbb{C}$. It is an antiderivative of f in the whole complex plane. Thus, one can apply FTC. The starting point of γ is $2 + \frac{1}{2}i$. The end point of γ is $1 + i$. You can use the command **ParametricPlot** or **Plot** to plot γ. (Please try it!)

$$\int_\gamma f(z)\,dz = F(1+i) - F\left(2 + i\frac{1}{2}\right) = \boxed{-5.43304 - 24.7084i}$$

$\square$

(e) $f(z) = z\text{Log}z$ where γ is the unit circle centered at the origin positively oriented.

Answer: By integration by part (in the manner of Problem 3c), we see that $F(z) = \frac{z^2}{2}\text{Log}z - \frac{z^2}{4}$ is an antiderivative of f in $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$. Because γ doesn't lie entirely in $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$, we cut γ with a small precision. Let γ_ϵ be the circle from the point $A = e^{i(-\pi+\epsilon)}$ to the point $B = e^{i(\pi-\epsilon)}$. We can use the FTC for the path γ_ϵ. Then

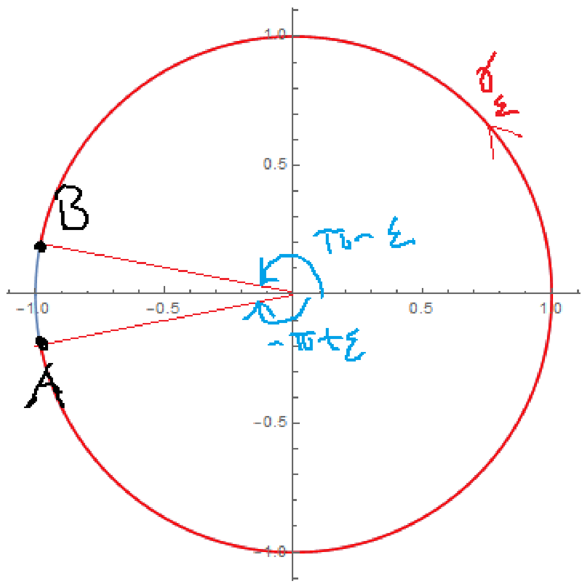


Figure 4:

$$\int_\gamma z\,\text{Log}\,z dz = \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} z\,\text{Log}\,z dz = \lim_{\varepsilon\to 0}\left(F\left(e^{i(\pi-\varepsilon)}\right) - F(e^{i(-\pi+\varepsilon)})\right) = \cdots$$

$\square$

(f) $f(z) = z^i$ where γ is the unit circle centered at the origin positively oriented.

Answer: Note that $f(z)$ has an antiderivative $F(z) = \frac{1}{i+1}z^{i+1}$ in the region $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$. Because γ doesn't lie entirely in this region, we cut γ the same way as Part (e).

$\square$

(g) $f(z) = \frac{z^2+1}{z+2}$ and γ is the unit circle centered at the origin negatively oriented.

Answer: $f(z)$ is holomorphic in the region enclosed by the unit circle, a simple closed curve. Then Cauchy-Goursat Theorem yields

$$\int_\gamma \frac{z^2+1}{z+2}\,dz = \boxed{0}$$

. $\square$

(h) $f(z) = \sin z$ and γ is the unit circle centered at the origin negatively oriented.

Answer: $\sin z$ is an entire function, thus Cauchy-Goursat Theorem implies

$$\int_\gamma \sin z \, dz = \boxed{0}$$

since γ is a simple closed curve. □

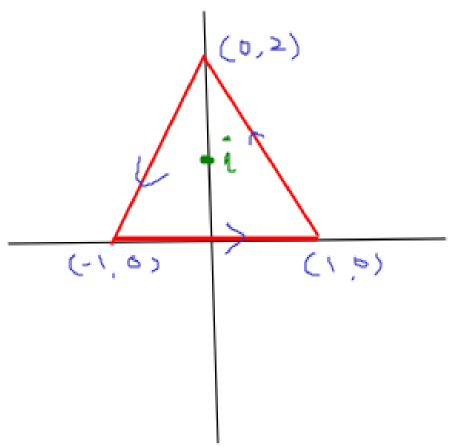
(i) $f(z) = \frac{e^z}{z^2+1}$ and γ is the triangle with vertices at $(-1,0), (1,0), (0,2)$ positively oriented.

Answer: Rewrite $f(z)$ as

$$\frac{e^z/(z+i)}{z-i} = \frac{g(z)}{z-i}$$

where $g(z) = \frac{e^z}{z+i}$. We see that g is holomorphic inside the triangle. This allows us to use Cauchy's Integral Formula

$$\int_\gamma f(z)\, dz = \int_\gamma \frac{g(z)}{z-i}\, dz = 2\pi i g(i) = \boxed{\pi e^i}$$



□

(j) $f(z) = \frac{e^z}{(z^2+1)^2}$ and γ is the circle with radius 2 centered at the origin negatively oriented.

Answer: We can rewrite

$$f(z) = \frac{e^z}{(z-i)^2(z+i)^2}$$

f is holomorphic everywhere except at $\pm i$. Because γ encloses two singular points, we need to split γ so that each curve encloses only onw singular point. Let us split γ the way indicated on the picture. Note that γ_1 and γ_2 are negatively oriented. Then

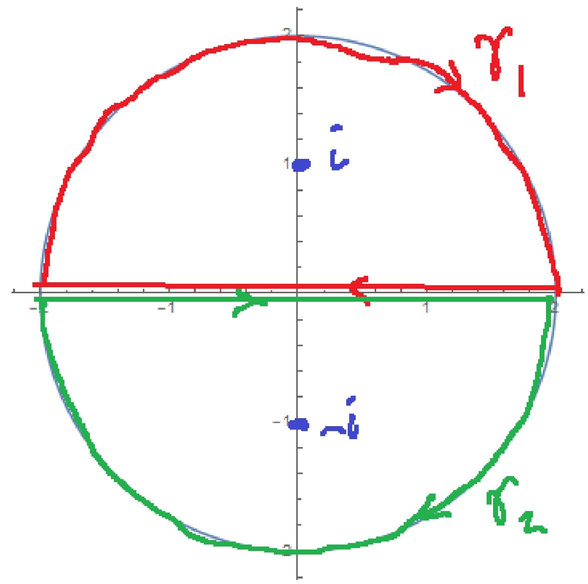


Figure 5:

$$\int_\gamma \frac{e^z}{(z-i)^2(z+i)^2}dz = \int_{\gamma_1} + \int_{\gamma_2} = \int_{\gamma_1} \frac{e^z/(z+i)^2}{(z-i)^2}dz + \int_{\gamma_2} \frac{e^z/(z-i)^2}{(z+i)^2}dz$$

Put $g(z) = e^z/(z+i)^2$ and $h(z) = e^z/(z-i)^2$. By the general Cauchy's Integral Formula,

$$\int_{\gamma_1} \frac{g(z)}{(z-i)^2}dz = -2\pi i g'(i) = \left(-\frac{1}{2} + \frac{i}{2}\right) e^i \pi.$$

$$\int_{\gamma_1} \frac{h(z)}{(z+i)^2}dz = -2\pi i h'(-i) = \left(\frac{1}{2} + \frac{i}{2}\right) e^{-i} \pi.$$

Therefore,

$$\int_\gamma \frac{e^z}{(z^2+1)^2}\,dz = \boxed{\left(-\frac{1}{2} + \frac{i}{2}\right) e^i \pi + \left(\frac{1}{2} + \frac{i}{2}\right) e^{-i} \pi}$$

□

(k) $f(z) = \frac{1}{z^2+z+1}$ and γ is the circle with radius 2 centered at the origin positively oriented.

Answer: f is holomorphic everywhere except at z's such that $z^2+z+1=0$. Solutions to this equation are $z_1 = -1/2 - \sqrt{3}i/2$ and $z_2 = -1/2 + \sqrt{3}i/2$. Please sketch them! We then split γ into two curves like in Part (j) for the same reason (except that both of them are positively oriented). Use Cauchy's Integral Formula

$$\int_\gamma f(z)dz = \int_\gamma \frac{1}{(z-z_1)(z-z_2)}dz = \int_{\gamma_1} \frac{1/(z-z_1)}{z-z_2}dz + \int_{\gamma_2} \frac{1/(z-z_2)}{z-z_1}dz$$

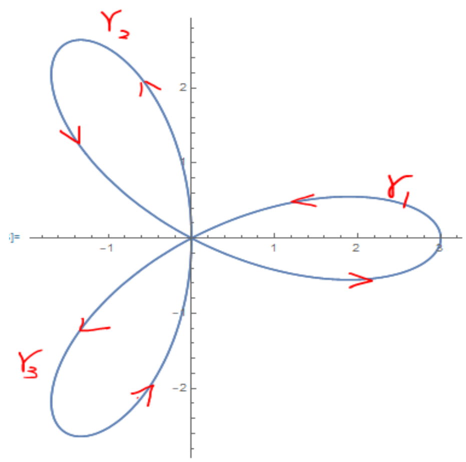
$$= 2\pi i \frac{1}{z_2 - z_1} + 2\pi i \frac{1}{z_1 - z_2} = 0.$$

$\square$

(l) $f(z) = \frac{1}{(z^2+2z+2)(z-1)}$ and γ is parametrized by

$$\begin{cases} x(t) = 3\cos t \cos 3t, \\ y(t) = 3\sin t \cos 3t \end{cases} \quad t \in [0, 2\pi]$$

Answer: First, we split the curve γ into three simple curves as shown in the graph below



f is holomorphic everywhere except at $z_1 = 1$, $z_2 = -1+i$, $z_3 = -1-i$. Use the same method as in Part (j) and (k), we get

$$\int_\gamma f(z)\, dz = \boxed{0}$$

$\square$

Exercise 5. To each of the following complex functions f,

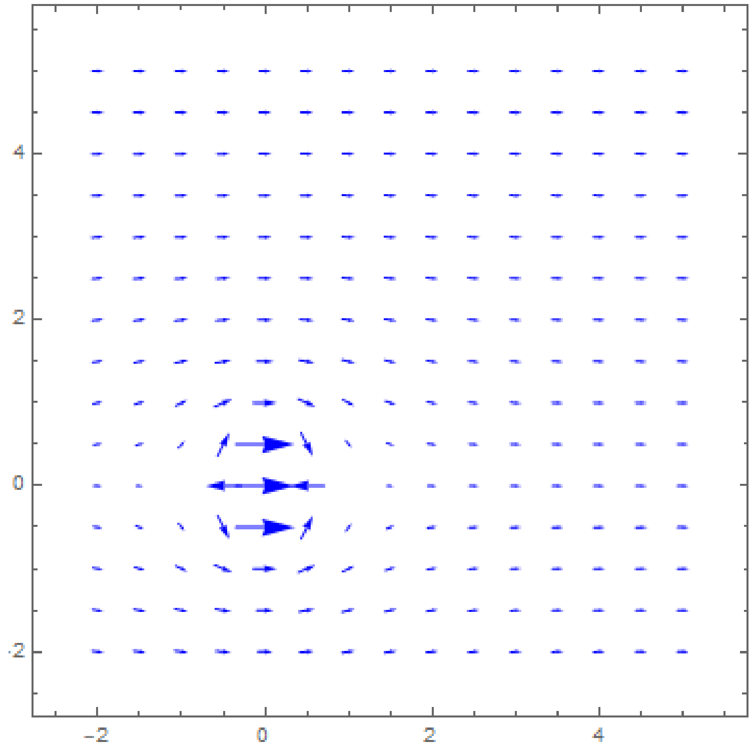
1. $f(z) = 1 - \frac{1}{z^2}$
2. $f(z) = z - \frac{1}{z^3}$
3. $f(z) = \frac{i}{\sqrt{1-z^2}}$

answer the following questions:

(a) Plot the Polya vector field associated with f. (This is the velocity field of an ideal flow.)

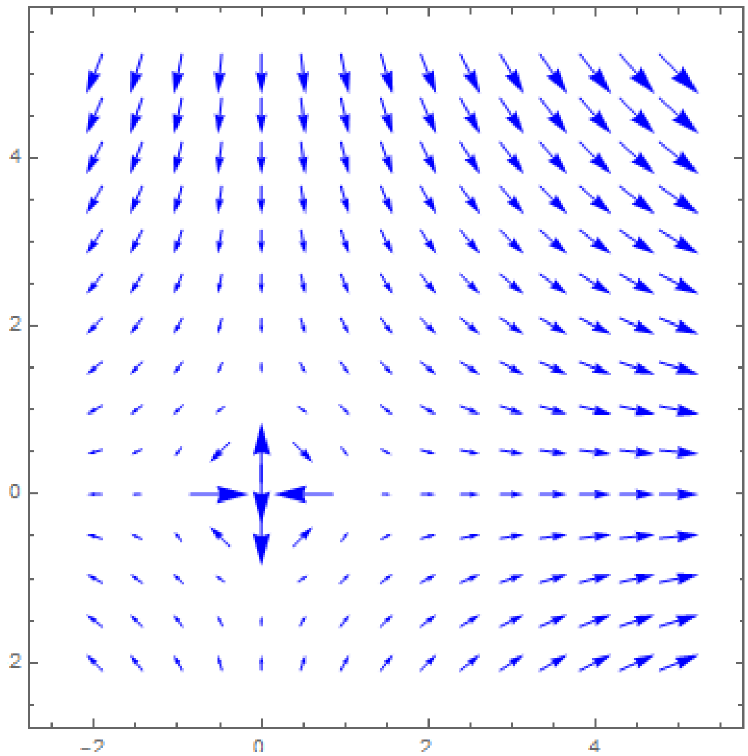
Answer: (1) $f(z) = 1 - \frac{1}{z^2}$:

```
    f[z_] := 1-z^(-2)
VectorPlot[{Re[f[x + I*y]], -Im[f[x + I*y]]}, {x, -2, 5}, {y, -2, 5},
VectorStyle -> Blue]
```



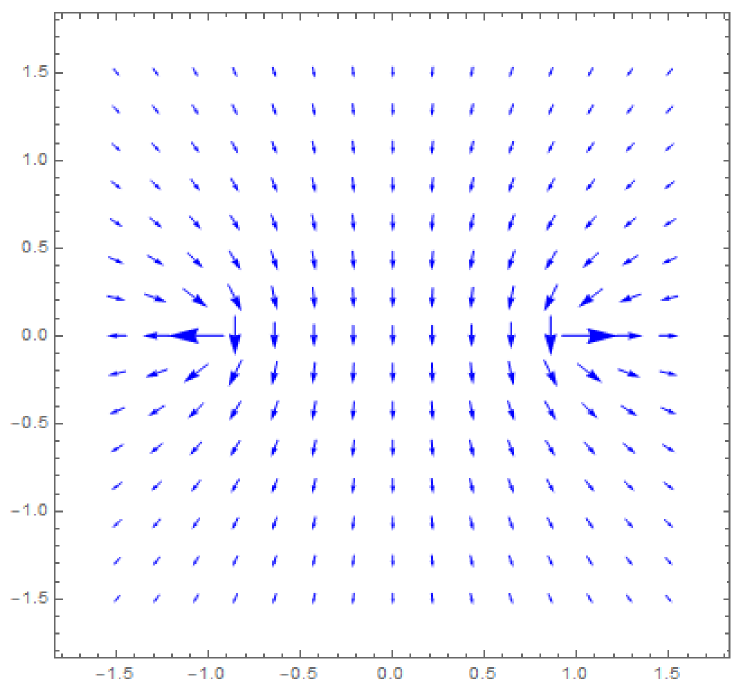
(2) $f(z) = z - \frac{1}{z^3}$:

```
   f[z_] := z-z^(-3)
VectorPlot[{Re[f[x + I*y]], -Im[f[x + I*y]]}, {x, -2, 5}, {y, -2, 5},
VectorStyle -> Blue]
```



(3) $f(z) = \frac{i}{\sqrt{1-z^2}}$:

```
  f[z_] := I/(Sqrt[1 - z^2])
VectorPlot[{Re[f[x + I*y]], -Im[f[x + I*y]]}, {x, -1.5, 1.5}, {y, -1.5, 1.5},
 VectorScale -> Automatic, VectorStyle -> Blue]
```



$\square$

(b) Determine a complex potential Φ of the flow.

Answer: (1) $z + \frac{1}{z}$ is an antiderivative of $1 - \frac{1}{z^2}$. Thus,

$$\Phi(z) = x + iy + \frac{1}{x+iy} = \left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$$

is a complex potential of $f(z) = 1 - \frac{1}{z^2}$.

(2) Similarly, $\frac{z^2}{2} + \frac{1}{2z^2}$ is an antiderivative of $z - \frac{1}{z^3}$. So,

$$\Phi(z) = \frac{(x+iy)^2}{2} + \frac{1}{2(x+iy)^2} = \frac{x^2-y^2}{2}\left(1 + \frac{1}{(x^2+y^2)^2}\right) + ixy\left(1 - \frac{1}{(x^2+y^2)^2}\right)$$

(3)

$$\Phi(z) = i\,\mathrm{Arc}\sin z$$

$\square$

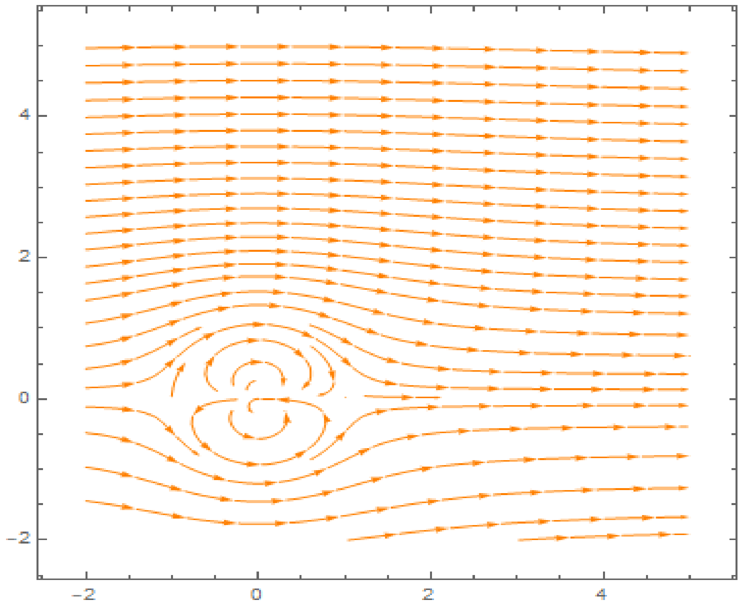
(c) Write the equation of streamlines of the flow. Then plot the streamlines.

Answer: (1) The imaginary part of the complex potential Φ is $\psi(x,y) = y - \frac{y}{x^2+y^2}$. Because a streamline is a level set of ψ, we conclude that the equation of the streamlines is

$$y - \frac{y}{x^2+y^2} = C.$$

We can plot the streamlines by Mathematica as follows.

```
    f[z_] := 1 - z^(-2)
StreamPlot[{Re[f[x + I*y]], -Im[f[x + I*y]]}, {x, -2, 5}, {y, -2, 5},
 StreamStyle -> Orange]
```

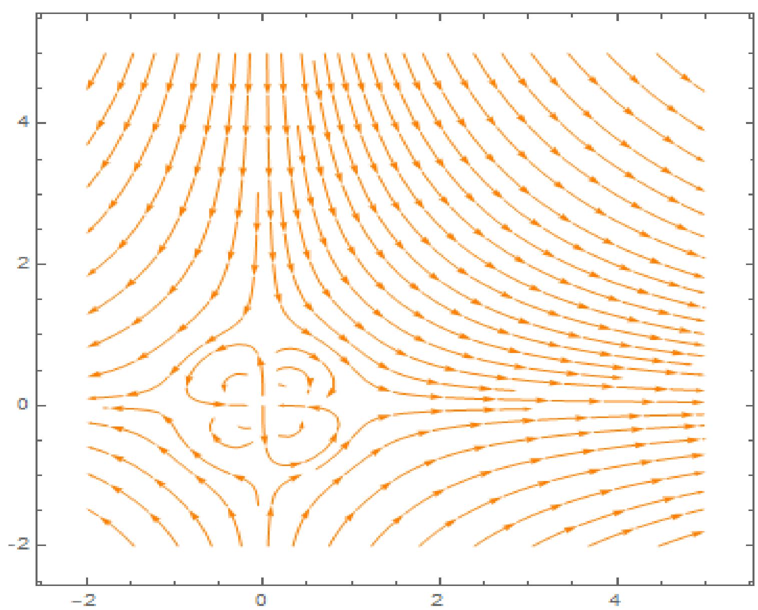


(2) The imaginary part of the complex potential Φ is $\psi(x,y) = xy\left(1 - \frac{1}{(x^2+y^2)^2}\right)$. Because a streamline is a level set of ψ, we conclude that the equation of the streamlines is

$$xy\left(1 - \frac{1}{(x^2+y^2)^2}\right) = C.$$

We can plot the streamlines by Mathematica as follows.

```
    f[z_] := z - z^(-3)
StreamPlot[{Re[f[x + I*y]], -Im[f[x + I*y]]}, {x, -2, 5}, {y, -2, 5},
 StreamStyle -> Orange]
```



(3) Let us put $w = \text{Arcsin}z = a + ib$. Here a and b are functions of x and y. Then $\Phi(z) = iw = -b + ia$. The imaginary of Φ is $\psi = a$. A streamline is a level set of ψ, which has equation $\psi(x, y) = C$. Thus, on each streamline we have $a = C$. There is no constraint on b. Let us write C or a and t for b to emphasize the fact that a is a constant and b is arbitrary. Then

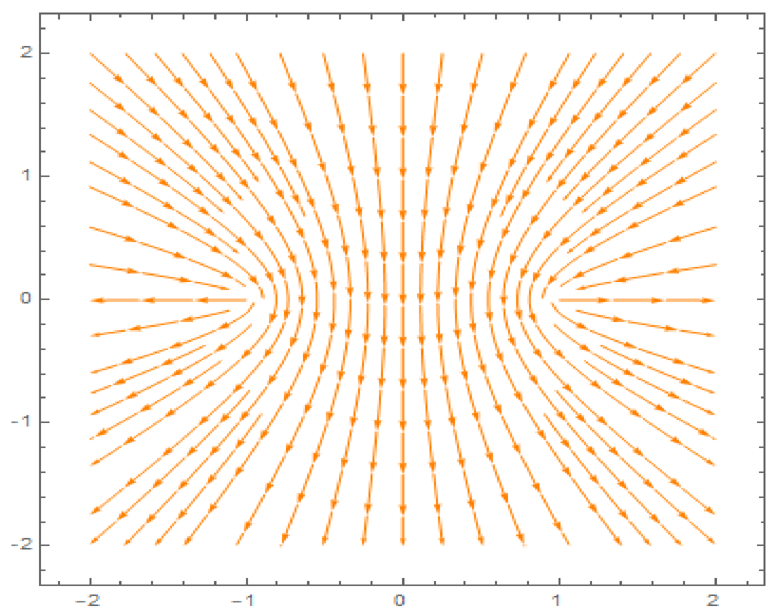
$$w = \text{Arcsin}z = C + it$$

We have

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{i(C+it)} - e^{-i(C+it)}}{2i} = \sin C \cosh(t) + i \cos C \sinh(t).$$

Therefore, the parametric equation (in complex form) of each streamline is

$$z(t) = \sin C \cosh(t) + i \cos C \sinh(t).$$



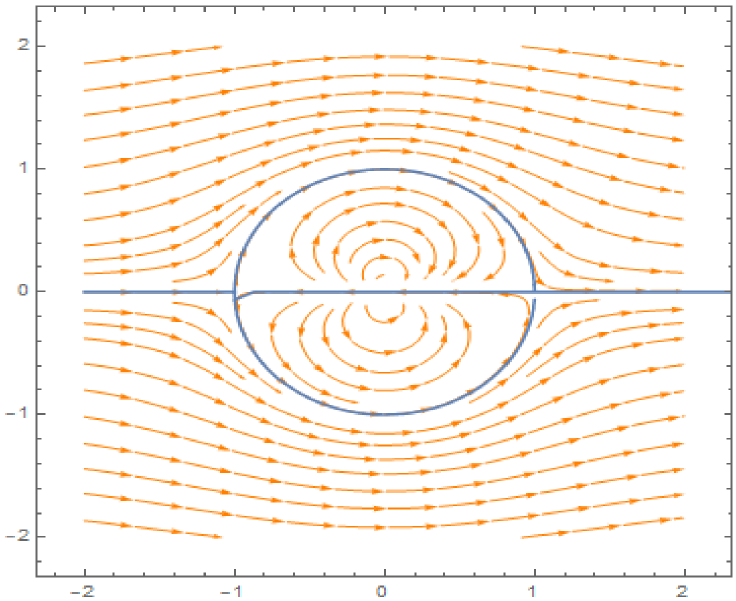
□

(d) Imagining that the flow you just plotted is a physical flow, can you determine the physical boundaries (i.e. walls and rigid obstacles) of the flow?

Answer: (1) For the function $f(z) = 1 - \frac{1}{z^2}$, we can see from the graph below that the physical boundaries are x-axis and the unit circle centered at the origin. This is the streamline with $C = 0$. Indeed, the equation

$$\psi(x, y) = y - \frac{y}{x^2 + y^2} = 0$$

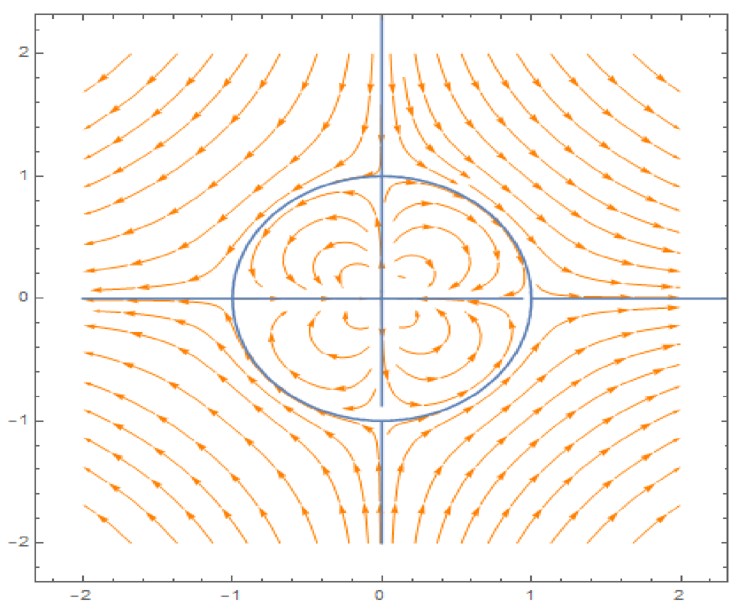
gives $y = 0$ (the horizontal line) and $x^2 + y^2 = 1$ (the unit circle).



(2) For the function $f(z) = z - \frac{1}{z^3}$, we can see from the graph below that the physical boundaries are: x-axis, y-axis, and the unit circle centered at the origin. This is the streamline with $C = 0$. Indeed, the equation

$$\psi(x, y) = xy\left(1 - \frac{1}{(x^2 + y^2)^2}\right) = 0$$

gives $x = 0$ (the vertical line), $y = 0$ (the horizontal line) and $x^2 + y^2 = 1$ (the unit circle).



(3) For the function $f(z) = \frac{i}{\sqrt{1-z^2}}$, we can see from the graph below that the physical boundaries the line $(-\infty, -1) \cup (1, \infty)$ on the real line. The flow is running through a hole on a punctured wall. The wall is the whole real line. The hole is at the interval $(-1, 1)$.

The punctured wall is made up two streamlines corresponding to $C = \pm\pi/2$. Indeed, the equation

$$z(t) = \sin\left(\frac{\pi}{2}\right)\cosh(t) + i\cos\left(\frac{\pi}{2}\right)\sinh(t)$$

gives

$$z(t) = \cosh(t).$$

This is the the line $(1, \infty)$. The equation

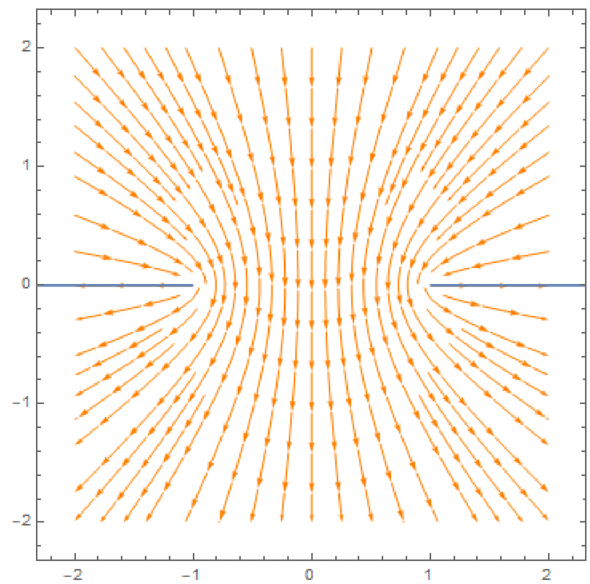
$$z(t) = \sin\left(-\frac{\pi}{2}\right)\cosh(t) + i\cos\left(-\frac{\pi}{2}\right)\sinh(t)$$

gives

$$z(t) = -\cosh(t).$$

This is the the line $(-\infty, -1)$.

```
f[z_] := I/Sqrt[1 - z^2]
p = StreamPlot[{Re[f[x + I*y]], -Im[f[x + I*y]]}, {x, -2, 2}, {y, -2,
   2}, StreamStyle -> Orange]
q1 = ParametricPlot[{Cosh[t], 0}, {t, -3, 3}]
q2 = ParametricPlot[{-Cosh[t], 0}, {t, -3, 3}]
Show[p, q1, q2]
```



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