# Homework 7

Due 6/5/2020

1. Let $f(z) = u(x,y) + iv(x,y)$ be a holomorphic function in region $D \subset \mathbb{C}$. Show that the vector field $P(x,y) = (u(x,y), -v(x,y))$ is incompressible and irrotational. In other words, show that $\mathrm{div}P = 0$ and $\mathrm{curl}P = 0$.
   *Note: You can review the definition of divergence and curl of a 2D vector field in the supplemental material "**Vector fields and complex integral**" posted on Canvas and the course website.*

2. Let $\Phi(z) = \phi(x,y) + i\psi(x,y)$ be an antiderivative of function $f(z) = u(x,y) + iv(x,y)$ in region $D \subset \mathbb{C}$. Show that at each point $(x,y) \in D$ the vector $P(x,y) = (u(x,y), -v(x,y))$ tangent to the level set of $\psi$ at passes through $(x,y)$. (For this reason, the level sets of $\psi$ are called the streamlines of vector field $P$.)
   *Hint: use the fact that the level sets of $\psi$ are perpendicular to the gradient vectors of $\psi$.*

3. Determine an antiderivative of the following function. Then identify the region where that antiderivative is holomorphic.
   (a) $f(z) = \frac{z}{z^2+1}$
   (b) $f(z) = z\sin(z^2+1)$
   (c) $f(z) = z^2 \mathrm{Log} z$

4. Evaluate the complex integrals $\int_\gamma f(z)$ where $f$ and $\gamma$ are given as follows. Clearly mention the method/theorem you use.
   (a) $f(z) = \bar{z}$ and $\gamma$ is the square with vertices at $(-1,-1)$, $(1,-1)$, $(1,1)$, $(-1,1)$ positively oriented.
   (b) $f(z) = |z|^2$ and $\gamma$ is the ellipse $x^2 + \frac{y^2}{4} = 1$ negatively oriented.
   (c) $f(z) = \frac{z^2+1}{z^3+3z+1}$ and $\gamma$ is the part of the parabola $y = x^2$ from $x = 0$ to $x = 2$.
   (d) $f(z) = z^3 e^z$ where $\gamma$ is the part of the hyperbola $y = \frac{1}{x}$ from $x = 2$ to $x = 1$.
   (e) $f(z) = z\mathrm{Log}z$ where $\gamma$ is the unit circle centered at the origin positively oriented.
   (f) $f(z) = z^i$ where $\gamma$ is the unit circle centered at the origin positively oriented.
   (g) $f(z) = \frac{z^2+1}{z+2}$ and $\gamma$ is the unit circle centered at the origin negatively oriented.
   (h) $f(z) = \sin z$ and $\gamma$ is the unit circle centered at the origin negatively oriented.
   (i) $f(z) = \frac{e^z}{z^2+1}$ and $\gamma$ is the triangle with vertices at $(-1,0)$, $(1,0)$, $(0,2)$ positively oriented.
   (j) $f(z) = \frac{e^z}{(z^2+1)^2}$ and $\gamma$ is the circle with radius 2 centered at the origin negatively oriented.
   (k) $f(z) = \frac{1}{z^2+z+1}$ and $\gamma$ is the circle with radius 2 centered at the origin positively oriented.
   (l) $f(z) = \frac{1}{(z^2+2z+2)(z-1)}$ and $\gamma$ is parametrized by

$$\begin{cases} x(t) = 3\cos t \cos 3t, \\ y(t) = 3\sin t \cos 3t \end{cases} \qquad t \in [0, 2\pi]$$

   *Hint:* split $\gamma$ into three simple curves. Pay attention to the orientation of each curve. Use Cauchy's Integral formula for each curve.

Before doing the following problem, please take a look at the supplemental material called "**Vector fields and complex integral**" posted on Canvas and course website. *Make sure to include the Mathematica codes and figures you use and some brief comments.*

5. To each of the following complex functions $f$,

   (1) $f(z) = 1 - \frac{1}{z^2}$

   (2) $f(z) = z - \frac{1}{z^3}$

   (3) $f(z) = \frac{i}{\sqrt{1-z^2}}$

   answer the following questions:

   (a) Plot the Polya vector field associated with $f$. (This is the velocity field of an ideal flow.)

   (b) Determine a complex potential $\Phi$ of the flow.

   (c) Write the equation of streamlines of the flow. Then plot the streamlines.

   (d) Imagining that the flow you just plotted is a physical flow, can you determine the physical boundaries (i.e. walls and rigid obstacles) of the flow?