

Lecture 2

Wednesday, April 1, 2020

We know how to solve for roots of first degree polynomials: given a, b , the equation $ax+b=0$ has a unique solution $x = -\frac{b}{a}$.

If a and b are integers, then x is not necessarily an integer. In fact, by trying to solve for this equation, e.g. $2x-1=0$, people discovered the rational numbers. The notation $\frac{1}{2}$ represents the root x of the equation $2x-1=0$. We often use the notation $\frac{1}{2}$ comfortably in computation as if it were integers. For example,

$$\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

Solving the equation $ax^2+bx+c=0$ is more complicated. One can divide both sides by a :

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

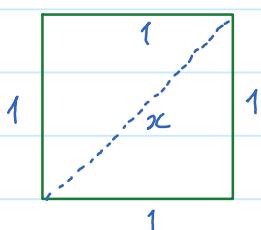
Then complete the square $\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$.

Then write $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$.

From here we can find x in terms of a, b, c . The quadratic equation $ax^2+bx+c=0$ with rational coefficients a, b, c may not have rational roots. For example, there is no rational number x such that $x^2=2$ (interesting exercise). However, there should be such a number: consider a square with sidelength equal to 1. Let x be the length of the

diagonal. Then by Pythagorean theorem we have

$$x^2 = 1^2 + 1^2 = 2.$$



One can of course measure approximately the diagonal by a ruler. Thus, the real numbers need to be introduced to represent such a quantity. The notation $\sqrt{2}$ represents the positive root of the equation $x^2 - 2 = 0$. We often use the notation $\sqrt{2}$ comfortably in computation as if it were integers or rational numbers. The only rule we need to know about $\sqrt{2}$ is that it is positive and its square is equal to 2. For example,

$$(1 + \sqrt{2})^2 = (1 + \sqrt{2})(1 + \sqrt{2}) = 1 + 1\sqrt{2} + 1\sqrt{2} + \underbrace{\sqrt{2} \cdot \sqrt{2}}_2 = 3 + 2\sqrt{2}.$$

A natural question is that: will we discover a new (richer) system of numbers by trying to solve the cubic equation

$$ax^3 + bx^2 + cx + d = 0 \quad ?$$

Around 1545, there were a group of Italian mathematicians who put efforts in solving this equation. Among them were G. Cardano, N. Tartaglia, S. Ferro. Their method is as follows. First, the cubic equation above can be converted to the form

$$y^3 + py + q = 0. \quad (*)$$

by putting $y = x + \frac{b}{3a}$. Now given p and q , how do we solve $(*)$ for y ? Cardano's idea is to split $y = u + v$ where u and v are to be determined. Then $(*)$ becomes

$$\underbrace{(u+v)^3}_{{u}^3+{v}^3+3{u}^2v+3uv^2} + p(u+v) + q = 0.$$

$$\underbrace{{u}^3+{v}^3+3uv(u+v)}_{u^3+v^3+3uv(u+v)}$$

Equivalently,

$$u^3 + v^3 + (3uv + p)(u+v) + q = 0.$$

We have one equation to solve for two unknowns u and v . The main trick is to impose a condition on u and v , namely $3uv + p = 0$. Then we have a system of two equations to solve for two unknowns.

$$\begin{cases} 3uv + p = 0 \\ u^3 + v^3 + q = 0 \end{cases}$$

We can rewrite this system as

$$\begin{cases} uv^3 = -\frac{p^3}{27} \\ u^3 + v^3 = -q \end{cases}$$

Put $t = u^3$ and $s = v^3$. Then we know the sum and the product of t and s .

Then t and s solve the equation

$$z^2 + qz - \frac{p^3}{27} = 0.$$

We know how to solve this quadratic equation:

$$z = \frac{1}{2} \left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}} \right).$$

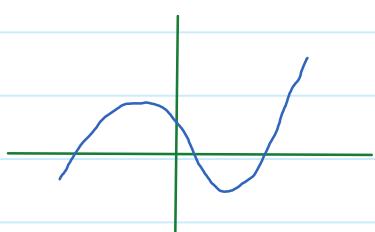
The plus sign gives us t , the minus sign s (or vice versa, which doesn't matter). From here one can find u and v . Then we get $y = u+v$.

The only issue in this approach is: what if the discriminant

$$\Delta = q^2 + \frac{4p^3}{27}$$

is negative? Consider, for example, the equation $y^3 - 3y + 1 = 0$.

Here $p = -3$, $q = 1$ and $\Delta = -3 < 0$. One can't take square root of -3 to get a real number. The function $f(y) = y^3 - 3y + 1$ has three real roots (see the graph). We should allow $\sqrt{-3}$ to have a meaning, although it is not a real number. Formally, one can write



$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1}.$$

Once $\sqrt{-1}$ is considered a number then $\sqrt{-3}$ is a number given by $\sqrt{-3} = \sqrt{3}\sqrt{-1}$. Let us regard $\sqrt{-1}$ as a number, keeping in mind that its square is equal to -1 . Then we can

Perform arithmetic operations with this new number. For example,

$$(1 + \sqrt{-1})^2 = (1 + \sqrt{1})(1 + \sqrt{-1}) = 1 + \sqrt{-1} + \underbrace{\sqrt{-1}}_{-1} + (\sqrt{-1})^2 = 2\sqrt{-1}.$$

There is an inconvenience with the notation. If we write $a = \sqrt{-1}$ then $a^2 = -1$. However, $(-a)^2 = (-1)^2 a^2 = a^2 = -1$. If we only described $a = \sqrt{-1}$ as a number whose square is equal to -1 then we cannot distinguish a from $-a$. In other words, the notation $\sqrt{-1}$ seems to be ambiguous.

Euler suggested the use of symbol i for $\sqrt{-1}$. Then there are two square roots of -1 , namely i and $-i$. From now, i is adopted into the family of numbers. The addition, multiplication, ... are performed naturally as with real numbers.

Def:

The set $\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$ is called the set of complex numbers.

The addition, subtraction, multiplication, division are defined as follows:

$$\begin{aligned} (a+bi) + (c+di) &\stackrel{\text{def}}{=} (a+c) + i(b+d) && \text{addition and subtraction} \\ (a+bi) - (c+di) &\stackrel{\text{def}}{=} (a-c) + i(b-d) && \text{are performed} \\ &&& \text{component-wise.} \end{aligned}$$

$$\begin{aligned} (a+bi)(c+di) &\stackrel{\text{def}}{=} ac + ibc + iad + \underbrace{i^2 bd}_{-1} \\ &= (ac - bd) + i(bc + ad). \\ \frac{a+bi}{c+di} &\stackrel{\text{def}}{=} \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac + bd + i(bc - ad)}{c^2 - d^2 i^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i \cdot \frac{bc - ad}{c^2 + d^2} \end{aligned}$$

For complex number $z = a + bi$ with $a, b \in \mathbb{R}$, a is said to be the real part, and b the imaginary part of z .

$a = \operatorname{Re}(z)$, $b = \operatorname{Im}(b)$. [Some textbooks don't use the parentheses.]

We see that the sum, difference, multiple, quotient of two complex numbers are still complex numbers. This gives \mathbb{C} a field structure. (We won't go into the detail of what an algebraic field is.)

We know how to exponentiate a real number. Given a real number x , how do we compute e^x ? This can be done approximately by hand.

We know that e^x has a Taylor expansion :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

One can compute a truncation of the right hand side by hand because there involve only the basic arithmetic operators. But how do we exponentiate the complex numbers? For example, how do we find (or define) $e^{i\theta}$?

We will discuss this issue next time.