Worksheet

5/15/2020

For each function $f(z)$ given below, find an antiderivative $F(z)$. Determine the region where F is holomorphic. Determine the region where F is holomorphic.

(a) $f(z) = z\text{Log}\, z$

$I = \int z \,\text{Log}\, z \, dz$

Let $\quad u = \text{Log}\, z$

$\quad\quad dv = z\, dz$

Then $\quad du = \frac{1}{z} dz \quad$ and $\quad v = \frac{z^2}{2}$.

By Integration by part,

$$I = \int u\, dv = uv - \int v\, du = \frac{z^2}{2} \text{Log}\, z - \int \frac{z^2}{2} \frac{1}{z} dz$$

$$= \frac{z^2}{2} \text{Log}\, z - \int \frac{z}{2} dz$$

$$= \frac{z^2}{2} \text{Log}\, z - \frac{z^2}{4} + C$$

An antiderivative of $f(z)$ is

$$F(z) = \frac{z^2}{2} \text{Log}\, z - \frac{z^2}{4}.$$

F is holomorphic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

(b) $f(z) = \frac{z}{z^2+1}$

$$I = \int \frac{z}{z^2+1}\, dz$$

Put $u = z^2+1$. Then $du = 2z\,dz$.

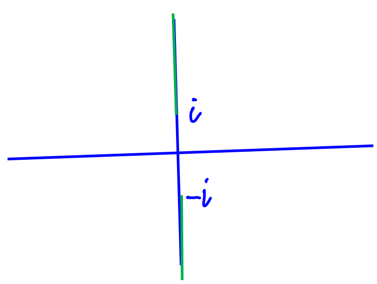
$$I = \int \frac{1}{2} \frac{du}{u} = \frac{1}{2} \int \frac{1}{u}\, du$$

There are many antiderivatives of $\frac{1}{u}$. One of them is $\operatorname{Log} u$.

$$I = \frac{1}{2} \operatorname{Log}(u) = \frac{1}{2} \operatorname{Log}(z^2+1).$$

An antiderivative of $f(z)$ is

$$F(z) = \frac{1}{2} \operatorname{Log}(z^2+1)$$



In HW 5, we showed that $\operatorname{Log}(z^2+1)$ is continuous everywhere except on the line

$$\begin{cases} x = 0 \\ y \geq 1 \end{cases} \text{ and } \begin{cases} x = 0 \\ y \leq -1. \end{cases}$$

Although f is continuous on $\mathbb{C}\setminus\{\pm i\}$, it doesn't have a an antiderivative on $\mathbb{C}\setminus\{\pm i\}$.

Theorem: If G is a simply connected and f is holomorphic on G then f has an antiderivative on G.

We will show this theorem later in the course.