Worksheet

5/8/2020

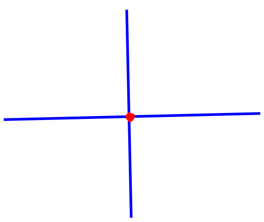
1. Find the region where function $f(z) = 1/z$ is holomorphic using the methods:

   (a) Definition of derivative $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

$$\frac{f(z+h)-f(z)}{h} = \frac{\frac{1}{z+h} - \frac{1}{z}}{h} = \frac{\frac{-h}{(z+h)z}}{h} = -\frac{1}{(z+h)z}.$$

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = -\frac{1}{z^2}$$

Therefore, f is differentiable at every $z \in \mathbb{C}\setminus\{0\}$.



f is holomorphic on $\mathbb{C}\setminus\{0\}$.

$$f'(z) = -\frac{1}{z^2}.$$

   (b) Differentiation rules: addition, multiplication, composition, etc.

Apply the quotient rule:

$$f'(z) = \left(\frac{1}{z}\right)' = \frac{1'z - 1z'}{z^2} = \frac{-1}{z^2} \quad \text{for any } z \neq 0.$$

(c) Cauchy–Riemann equations.

Write $z = x+iy$.

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} + i\underbrace{\frac{-y}{x^2+y^2}}_{v(x,y)}$$

We have

$$u_x = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_y = \frac{(-1)(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = \frac{-(-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

We see that

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{for any } (x,y) \neq (0,0).$$

Therefore, f is differentiable everywhere in $\mathbb{C}\setminus\{0\}$.

f is holomorphic on $\mathbb{C}\setminus\{0\}$.

$$f'(z) = u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}.$$

$\left(\text{which is equal to } \frac{-1}{(x+iy)^2} = \frac{-1}{z^2}\right)$