## Group work 1

Due 1/18/2019

## Name:

Instructions: Show your work. In problems that require Mathematic (for example Problem 5 , Part (vi) and (xi)), write one or two commands that best illustrate how you get the answer. You are not required to write all the codes. Circle your final answers. The assignment has 6 pages.

1. Express the following sums using sigma notation. (Answers are not unique.)

3 pt .(i) $1 \cdot 3+2 \cdot 4+3 \cdot 5+\ldots+98 \cdot 100$

$$
\sum_{k=1}^{98} k(k+2), \text { or } \sum_{k=2}^{99}(k-1)(k+1), \cdots
$$

3 (ii) $\frac{1}{n} \sin \left(\frac{2 \pi}{n}\right)+\frac{1}{n} \sin \left(\frac{4 \pi}{n}\right)+\frac{1}{n} \sin \left(\frac{6 \pi}{n}\right)+\ldots+\frac{1}{n} \sin \left(\frac{n 2 \pi}{n}\right)$

$$
\sum_{k=1}^{n} \frac{1}{n} \sin \left(\frac{2 \pi k}{n}\right)
$$

2. Fill in the blank spaces to obtain a correct equation.

3 (i) $\sum_{k=0}^{n-1} k=\sum_{j=1}^{n}(\mathrm{j}-1)$

3 (ii) $\sum_{k=2}^{n} k^{2}=\sum_{j=0}^{n-2}(j+2)^{2}$
3. Given $\sum_{k=0}^{n-1} k=\frac{n(n-1)}{2}$. Compute

$$
3 \text { (i) } \sum_{k=1}^{n} k=1+2+\cdots+n=\frac{0+1+\cdots+(n-1)}{\sum_{k=0}^{n-1} k}+n=\frac{n(n-1)}{2}+n=\frac{n(n+1)}{2}
$$

4 (ii)

$$
\begin{aligned}
\sum_{k=1}^{2 n}(3 k+2) & =3(1)+2+3(2)+2+3(3)+2+\cdots+3(2 n)+2 \\
& =3(1+2+3+\ldots+2 n)+\underbrace{2+\cdots+2}_{2 n \text { terms }} \\
& =3 \sum_{k=1}^{2 n} k+2(2 n)
\end{aligned}
$$

In Part (i), we replace $n$ by $2 n: \quad \sum_{k=1}^{2 n} k=\frac{2 n(2 n+1)}{2}=n(2 n+1)$
Therefore $\sum_{k=1}^{2 n}(3 k+2)=3 n(2 n+1)+4 n$
4. Let $q$ be a positive number. Denote $S_{n}=1+q+q^{2}+\ldots+q^{n}$.

3 (i) Express $S_{n}$ using sigma notation.

$$
S_{n}=\sum_{k=0}^{n} q^{k}
$$

3
(ii) Find an explicit formula for $S_{n}$ in terms of $n$.

$$
\begin{aligned}
& q S_{n}=q^{1}+q^{2}+q^{3}+\ldots+q^{n}+q^{n+1} \\
& \ddots \ddots \ddots \ddots \\
& S_{n}=q^{0}+q^{1}+q^{2}+\cdots+q^{n-1}+q^{n} \\
& \underbrace{q_{n}-S_{n}}_{(q-1) S_{n}}=q^{n+1}-q^{0}=q^{n+1}-1
\end{aligned}
$$

Thus, $\quad s_{n}=\frac{q^{n+1}-1}{q-1}$
5. Let $f(x)=2^{x}-x$.


(ii) Partition $[0,2]$ into $n$ equal subintervals $\left[x_{0}, x_{1}\right],\left[x_{1}, x_{2}\right], \ldots,\left[x_{n-1}, x_{n}\right]$ with $0=x_{0}<$ 3 $x_{1}<\ldots<x_{n-1}<x_{n}=2$. Write an expression for $x_{k}$.

$$
\begin{array}{lllll}
0 & \frac{2}{n} & \frac{4}{n} & \frac{6}{n} & 2
\end{array} \quad x_{k}=\frac{2 k}{n}, \quad k=0,1,2, \ldots, n
$$

(iii) What is the width of each subinterval?

$$
\frac{2}{n}
$$

(iv) On each subinterval, we denote by $x_{k}^{*}$ a point chosen from the $k$ 'th subinterval $\left[x_{k}, x_{k+1}\right]$. The area of the narrow region between the graph of $f$ and $\left[x_{k}, x_{k+1}\right]$ is approximated by $f\left(x_{k}^{*}\right) \Delta x_{k}$, where $\Delta x_{k}$ is the width of the $k^{\prime}$ th subinterval. What is $x_{k}^{*}$ in the following cases?

- Left-point Riemann sum

$$
2 \quad x_{k}^{*}=x_{k}=\frac{2 k}{n}
$$

- Right-point Riemann sum

$$
2 \quad x_{k}^{*}=x_{k+1}=\frac{2(k+1)}{n}
$$

- Midpoint Riemann sum

$$
2 \quad x_{k}^{*}=\frac{x_{k}+x_{k+1}}{2}=\frac{\frac{2 k}{n}+\frac{2(k+1)}{n}}{2}=\frac{2 k+1}{n}
$$

(v) Write the left-point Riemann sum. Denote it by $I_{n}$.
$3 I_{n}=\sum_{k=0}^{n-1} f\left(x_{k}\right) \frac{2}{n}=\sum_{k=0}^{n-1}\left(2^{x_{k}}-x_{k}\right) \frac{2}{n}=\sum_{k=0}^{n-1}\left(2^{\frac{2 k}{n}}-\frac{2 k}{n}\right) \frac{2}{n}$
(vi) Use Mathematica to compute this sum for $n=5, n=50, n=500, n=5000$. Round-up 3 to 4 digits after the decimal point. Does the sequence $I_{n}$ seem to converge?

$$
\begin{aligned}
& \operatorname{Snm}\left[2^{n}(2 k / n)-2 k / n,\left\{k_{1}, 0, n-1\right\}\right] \\
& n=5 \ldots 2.1558 \\
& n=50 \ldots 2.3084 \\
& n=500 \ldots 2.3279 \\
& n=5000 \ldots 2
\end{aligned}
$$

(vii) Use Problem 3 and Problem 4 to compute an explicit formula for $I_{n}$ in term of $n$.

4

$$
I_{n}=\sum_{k=0}^{n-1}\left(2^{\frac{2 k}{n}}-\frac{2 k}{n}\right) \frac{2}{n}=\sum_{k=0}^{n-1}\left(2^{\frac{2 k}{n}} \frac{2}{n}-\frac{2 k}{n} \frac{2}{n}\right)=\underbrace{\sum_{k=0}^{n-1} 2^{\frac{2 k}{n}} \frac{2}{n}}_{A}-\underbrace{\sum_{k=0}^{n-1} \frac{2 k}{n} \frac{2}{n}}_{B}
$$

$B$ is easier to compute:

$$
\begin{aligned}
& B=\frac{4}{n^{2}} \sum_{k=0}^{n-1} k=\frac{4}{n^{2}} \frac{n(n-1)}{2}=\frac{2(n-1)}{n} \\
& A=\frac{2}{n} \sum_{k=0}^{n-1} 2^{\frac{2 k}{n}}=\frac{2}{n} \sum_{k=0}^{n-1}\left(2^{\frac{2}{n}}\right)^{k}
\end{aligned}
$$

I ut $q=2^{\frac{2}{n}}$. Then $A=\frac{2}{n} \sum_{k=0}^{n-1} q^{k}=\frac{2}{n} \frac{q^{n}-1}{9-1}=\frac{2}{n} \frac{\left(2^{\frac{2}{n}}\right)^{n}-1}{2^{\frac{2}{n}}-1}=\frac{6}{n\left(2^{\frac{2}{n}}-1\right)}$

Therefore,

$$
I_{n}=A-B=\frac{6}{n\left(2^{\frac{2}{n}}-1\right)}-\frac{2(n-1)}{n}
$$

(viii) Use the fact that $\lim _{x \rightarrow 0} \frac{2^{x}-1}{x}=\ln 2$ to compute $\lim _{n \rightarrow \infty} I_{n}$. This number is the exact area of 4 the region between the graph of $f$ and the $x$-axis.

Therefore, $\lim _{n \rightarrow \infty} I_{n}=\frac{3}{\ln 2}-2$
(ix) With the partition $0<x_{0}<x_{1}<\ldots<x_{n}=2$ as above, sketch a picture to illustrate 3 how the length of the graph of $f$ is approximated by a polyline.

(x) Write a sum that approximates the length of the graph of $f$. 3

$$
\underbrace{}_{x_{k+1}-x_{k}}\} f\left(x_{k+1}\right)-f\left(x_{k}\right)
$$

length of the kith segment

$$
=\sum_{k=0}^{n-1} \sqrt{\left(\frac{2(k+1)}{n}-\frac{2 k}{n}\right)^{2}+\left[\left(2^{\frac{2(k+1)}{n}}-\frac{2(k+1)}{2}\right)-\left(2^{\frac{2 k}{n}}-\frac{2 k}{n}\right)\right]^{2}}
$$

$$
\begin{aligned}
& I_{n}=\frac{6}{n\left(2^{\frac{2}{n}}-1\right)}-\frac{2(n-1)}{n}=\frac{\frac{6}{n}}{\underbrace{2^{\frac{2}{n}}-1}_{\text {is equal to }}}-\underbrace{\frac{2 n-2}{n}}_{\rightarrow 2}
\end{aligned}
$$

(xi) Use Mathematic to compute approximate length of the graph of $f$ for $n=10, n=50$, $3 n=100$. Round-up to 4 digits after the decimal point.

$$
\begin{gathered}
\operatorname{Sum}\left[\operatorname{Sart}\left[\left(\frac{2(k+1)}{n}-\frac{2 k}{n}\right)^{2}+\left(\left(2^{\frac{2(k+1)}{n}}-\frac{2(k+1)}{2}\right)-\left(2^{\frac{2 k}{n}}-\frac{2 k}{n}\right)\right)^{2}\right],\{k, 0, n-1\}\right] \\
n=10: 2.4600 \\
n=50: 2.4618 \\
n=100: 2.4619
\end{gathered}
$$

