

Group work 3

Due 2/1/2019

Total: 67 points

Name: _____

Instructions: Show your work. In problems that require Mathematica, write one or two commands that best illustrate how you get the answer. You are not required to write all the codes. Circle your final answers. The assignment has 6 pages.

1. Find a function $F(x)$ such that $F''(x) = 6x + 4$ and $F(0) = 1$, $F'(0) = 3$.

$$F'(x) = 3x^2 + 4x + C$$

4

$$\text{Plug } x=0: 3 = 3(0)^2 + 4(0) + C \Rightarrow C=3$$

$$F'(x) = 3x^2 + 4x + 3$$

$$\Rightarrow F(x) = x^3 + 2x^2 + 3x + D$$

$$\text{Plug } x=0: 1 = 0^3 + 2(0)^2 + 3(0) + D \Rightarrow D=1$$

$$\text{Conclusion: } F(x) = x^3 + 2x^2 + 3x + 1$$

2. Find the following indefinite integral:

$$\int (\sin 2x + \cos 2x) dx.$$

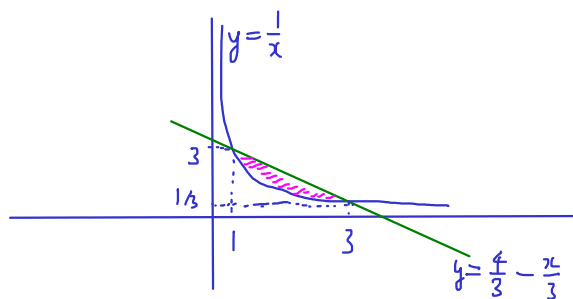
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(Your answer should contain an undetermined constant.)

$$-\frac{\cos 2x}{2} + \frac{\sin 2x}{2} + C$$

3. In this problem, we practice computing the area between two curves by definite integral.

- (i) Graph the functions $y = \frac{1}{x}$ and $y = \frac{4}{3} - \frac{x}{3}$ (on the same graph) in the first quadrant. Then shade the region R enclosed by these curves.



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- (ii) Find the intercepts of these two curves.

$$\text{Set } \frac{1}{x} = \frac{4}{3} - \frac{x}{3}.$$

$$\text{Multiply both sides by } 3x; \quad 3 = (4-x)x = 4x - x^2$$

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$$\text{Then } x^2 - 4x + 3 = 0$$

This quadratic polynomial has two roots $x_1 = 1$ and $x_2 = 3$

$$\text{For } x = 1, \quad y = \frac{1}{x} = 1$$

$$\text{For } x = 3, \quad y = \frac{1}{x} = \frac{1}{3}$$

Two intercepts are $(1, 1)$ and $(3, \frac{1}{3})$

- (iii) Express the area of R as a definite integral.

$$\int_1^3 \left(\frac{4}{3} - \frac{x}{3} - \frac{1}{x} \right) dx$$

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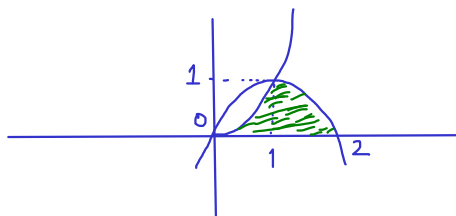
- (iv) Evaluate the area of R .

$$\begin{aligned} \left(\frac{4}{3}x - \frac{x^2}{6} - \ln|x| \right) \Big|_1^3 &= \left(\frac{4}{3}(3) - \frac{3^2}{6} - \ln 3 \right) - \left(\frac{4}{3} - \frac{1^2}{6} - \underbrace{\ln 1}_{=0} \right) \\ &= \frac{4}{3} - \ln 3 \end{aligned}$$

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4. Sometimes the region needs to be split into two (or more) pieces in a way so that the area of each piece is easy to compute.

- (i) Graph the functions $y = x^2$ and $y = 2x - x^2$ on the same graph. Then shade the region R whose boundary is made up by the curves $y = 0$, $y = x^2$ and $y = 2x - x^2$.



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- (ii) Find the intercepts of these two graphs.

$$\text{set } x^2 = 2x - x^2$$

$$\text{One gets } 2x(x-1) = 1, \text{ which gives } x = 0 \text{ and } x = 1$$

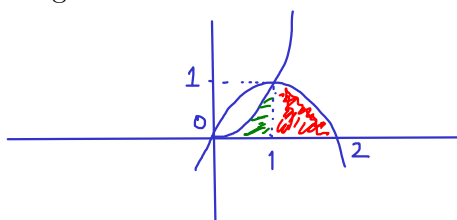
$$\text{For } x = 0, \quad y = x^2 = 0$$

$$\text{For } x = 1, \quad y = x^2 = 1$$

The two intercepts are $(0, 0)$ and $(1, 1)$.

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- (iii) Divide R into two subregions and express the area of R as the sum of two definite integrals.



$$\text{Area}(R) = \text{Area}(R_1) + \text{Area}(R_2)$$

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$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx$$

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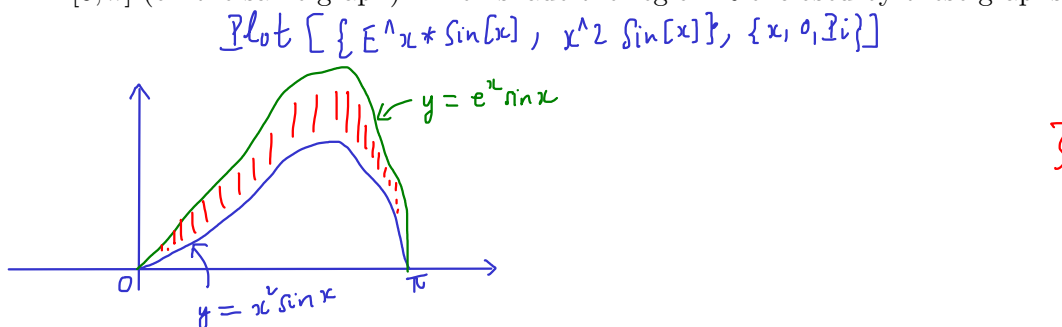
- (iv) Evaluate the area of R .

$$\begin{aligned} \text{Area}(R) &= \left. \frac{x^3}{3} \right|_0^1 + \left. \left(x^2 - \frac{x^3}{3} \right) \right|_1^2 = \frac{1}{3} - \left[\left(2^2 - \frac{2^3}{3} \right) - \left(1^2 - \frac{1^3}{3} \right) \right] \\ &= 1 \end{aligned}$$

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5. In the two previous exercises, one is able to compute definite integrals by the Fundamental Theorem of Calculus. Sometimes it is not easy to find antiderivatives. In such a case, one needs to approximate the definite integral using Riemann sum.

- (i) Use Mathematica to graph the functions $y = e^x \sin x$ and $y = x^2 \sin x$ on the interval $[0, \pi]$ (on the same graph). Then shade the region R enclosed by these graphs.



- (ii) Partition $[0, \pi]$ into 100 equal subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{99}, x_{100}]$. What is the width of each subinterval?

$$\frac{\pi}{100}$$

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- (iii) Determine $x_0, x_1, x_2, \dots, x_{100}$.

$$x_0 = 0$$

$$x_1 = \frac{\pi}{100}$$

$$x_2 = \frac{2\pi}{100}, \dots$$

$$x_k = \frac{k\pi}{100}$$

2

- (iv) Express the left-point Riemann sum (using the sigma summation notation) that approximates the area of R .

Function: $e^x \sin x - x^2 \sin x = f(x)$

$$\sum_{k=0}^{99} f\left(\underbrace{x_k}_{\frac{k\pi}{100}}\right) \underbrace{(x_{k+1} - x_k)}_{\frac{\pi}{100}} = \sum_{k=0}^{99} \left[e^{\frac{k\pi}{100}} \sin\left(\frac{k\pi}{100}\right) - \left(\frac{k\pi}{100}\right)^2 \sin\left(\frac{k\pi}{100}\right) \right] \frac{\pi}{100}$$

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- (v) Use Mathematica to evaluate this sum.

$$x[k] := k\pi/100$$

$$f[x] := E^x * \sin[x] - x^2 * \sin[x]$$

$$\text{Sum}[f[x[k]] * \pi/100, \{k, 0, 99\}]$$

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6. In some cases, the Fundamental Theorem of Calculus can be used to compute limits. Put

$$S_n = \sum_{k=1}^n \left(1 + \frac{k^2}{n^2}\right) \frac{2}{n}$$

- (i) S_n is a Riemann sum of a function f on an interval $[a, b]$. What is $f(x)$? (The answer is not unique.)

$$f(x) = 2(1+x^2) \quad 2$$

- (ii) Here n is a generic number assumed to be known. How many subintervals are there in the partition?

$$n \text{ (the numbers of terms in the summation)} \quad 2$$

- (iii) With this choice of function f , what is the interval $[a, b]$?

$$[0, 1] \quad 2$$

- (iv) What is the width of each subinterval?

$$\frac{1}{n} \quad 2$$

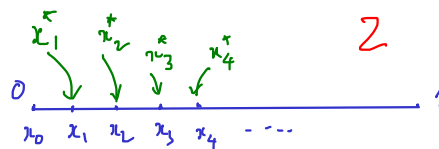
- (v) What are the gridpoints $a = x_0 < x_1 < x_2 < \dots < x_n = b$?

$$x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \dots$$

$$x_k = \frac{k}{n} \quad 2$$

- (vi) What are the sample points x_k^* ?

$$x_k^* = x_k = \frac{k}{n}$$



- (vii) Express $\lim_{n \rightarrow \infty} S_n$ as a definite integral.

S_n is a right point Riemann sum of $f(x)$ on the interval $[0, 1]$.

$$\int_0^1 2(1+x^2) dx \quad 3$$

(viii) Evaluate $\lim_{n \rightarrow \infty} S_n$.

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$$\int_0^1 (2+2x^2) dx = \left(2x + \frac{2x^3}{3}\right) \Big|_0^1 = 2 + \frac{2(1)^3}{3} = \frac{8}{3}$$

(ix) Use Mathematica to compute S_{150} . (You expect to get a value close to the limit found above.)

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$$n=150$$

$$\text{Sum} \left[\left(1 + k^2/n^2\right) \times \frac{2}{n}, \{k, 1, n\} \right]$$

$$2.6734$$