Group work 5

57 points in total

Due 3/8/2019

Name: ______________________________

Instructions: Show your work. Circle your final answers. The assignment has 6 pages.

1. Find the following integrals:

   (a)
   $$\int \frac{1}{1-x^2}dx$$
   Hint: use partial fraction decomposition.

1 pt $$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

Require : $1 = A(1+x) + B(1-x)$

2 pt { Plug $x = 1$: $A = \frac{1}{2}$

Plug $x = -1$: $B = \frac{1}{2}$ }

2 pt { $$\int \frac{1}{1-x^2}dx = \frac{1}{2}\int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{1}{1+x}dx = -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + C$$ }

   (b)
   $$\int \frac{1}{\cos x}dx$$
   Hint: use substitution $u = \sin x$.

2 pt { $$\int \frac{\cos x}{\cos^2 x}dx = \int \frac{du}{1-u^2} = -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C$$ }

(from part a)

2 pt { $$= -\frac{1}{2}\ln(1-\sin x) + \frac{1}{2}\ln(1+\sin x) + C$$ }

(c)

$$\int \sqrt{1+x^2}dx$$

Hint: use substitution $x = \tan u$.

2pt $\Big\{$ $dx = x' du = (1+\tan^2 u)\, du = (1+x^2) du$

$$\sqrt{1+x^2}\,dx = \sqrt{1+x^2}\,(1+x^2)\,du = (1+x^2)^{3/2} du = (1+\tan^2 u)^{3/2} du$$

$$= \frac{1}{(\cos^2 u)^{3/2}} du$$

2pt $\Big\{$ $= \frac{1}{\cos^3 u} du$

2pt

$$\int \sqrt{1+x^2}\,dx = \int \frac{1}{\cos^3 u} du = \underbrace{\int \frac{\cos u}{\cos^4 u} du}$$

Put $v = \sin u$.

Then $\cos^4 u = (\cos^2 u)^2$
$= (1-v^2)^2$

$$= \int \frac{dv}{(1-v^2)^2} = \int \frac{dv}{(1-v)^2(1+v)^2}$$

$$= \int \left(\frac{A}{1-v} + \frac{B}{(1-v)^2} + \frac{C}{1+v} + \frac{D}{(1+v)^2}\right) dx$$

. . . . .

(Sorry I didn't leave enough space)

(d)

$$\int_0^{\pi/2} x\cos 2x dx$$

Hint: use integration by parts.

2pt $\Big\{$

$u = x \quad \text{-- ---.-} \quad du = dx$

$dv = \cos 2x\, dx \quad \text{......} \quad v = \frac{1}{2}\sin 2x$

2pt

$$\int_0^{\pi/2} x\cos 2x\,dx = \underbrace{\frac{1}{2} x\sin 2x\Big|_0^{\pi/2}}_{=0} - \int_0^{\pi/2} \frac{1}{2}\sin 2x\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cos 2x\Big|_0^{\pi/2}$$

$$= \frac{1}{4}(\cos\pi - \cos 0)$$

$$= -\frac{1}{2}$$

(e)

$$\int_0^3 \frac{x}{\sqrt{x+1}}dx$$

Hint: use either substitution or integration by parts.

2pt $\Big\{$ $u = x+1 \; \ldots\ldots \; du = dx$

x	0	3
u	1	4

2pt

$$\int_1^4 \frac{u-1}{\sqrt{u}}du = \int_1^4 \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du = \left(\frac{2}{3}u^{3/2} - 2\sqrt{u}\right)\Big|_1^4$$

$$= \frac{2}{3}(4)^{3/2} - 2\sqrt{4} - \left(\frac{2}{3} - 2\right)$$

$$= \frac{8}{3}$$

(f)

$$\int e^x \cos^2 x dx$$

Hint: use the identity $\cos 2x = 2\cos^2 x - 1$.

2pt $\Big\{$ $\cos^2 x = \frac{1}{2}(\cos 2x + 1) = \frac{1}{2}\cos 2x + \frac{1}{2}$

2pt $\Big\{$ $\int e^x \cos^2 x dx \quad \int e^x \left(\frac{1}{2}\cos 2x + \frac{1}{2}\right)dx = \frac{1}{2}\underbrace{\int e^x \cos 2x dx}_{I} + \frac{1}{2}\underbrace{\int e^x dx}_{e^x + C}$

2pt $\Big\{$ To compute I, use integration by parts twice:

$u = e^x \; \ldots\ldots \; du = e^x dx$

$dv = \cos 2x dx \; \ldots\ldots \; v = \frac{1}{2}\sin 2x$

$$I = \frac{1}{2}e^x \sin 2x - \frac{1}{2}\underbrace{\int e^x \sin 2x dx}_{J}$$

2pt $\Big\{$ $u = e^x \; \ldots\ldots \; du = e^x dx$

$dv = \sin 2x dx \; \ldots\ldots \; v = -\frac{1}{2}\cos 2x$

2pt $\Big\{$ $J = -\frac{1}{2}e^x \cos 2x + \frac{1}{2}\int e^x \cos 2x dx = -\frac{1}{2}e^x \cos 2x + \frac{1}{2}I$

$$I = \frac{1}{2}e^x \sin 2x - \frac{1}{2}\left(-\frac{1}{2}e^x \cos 2x + \frac{1}{2}I\right)$$

$$= \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}I$$

$$I = \frac{4}{5}\left(\frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x\right)$$

(g)

$$I = \int_0^{\pi} e^x \cos 2x dx$$

Hint: use integration by parts twice.

2pt
$$u = e^x \quad \ldots\ldots \quad du = e^x dx$$
$$v = \cos 2x \quad \ldots\ldots \quad v = \frac{1}{2}\sin 2x$$

2pt
$$I = \underbrace{\frac{1}{2} e^x \sin 2x \Big|_0^{\pi}}_{=0} - \frac{1}{2} \underbrace{\int_0^{\pi} e^x \sin 2x dx}_{J} = -\frac{1}{2} J \qquad (*)$$

2pt
$$u = e^x \quad \ldots\ldots \quad du = e^x dx$$
$$dv = \sin 2x dx \quad \ldots\ldots \quad v = -\frac{1}{2}\cos 2x$$

2pt
$$J = \underbrace{-\frac{1}{2} e^x \cos 2x \Big|_0^{\pi}} + \frac{1}{2} \int_0^{\pi} e^x \cos 2x dx = -\frac{1}{2} e^{\pi} + \frac{1}{2} + \frac{1}{2} I$$
$$-\frac{1}{2} e^{\pi} \cos 2\pi + \frac{1}{2} e^0 \cos 0$$
$$= -\frac{1}{2} e^{\pi} + \frac{1}{2}$$

Substitute this J into (*):

2pt
$$I = -\frac{1}{2}\left(-\frac{1}{2} e^{\pi} + \frac{1}{2} + \frac{1}{2} I\right) = -\frac{1}{2}\left(-\frac{1}{2} e^{\pi} + \frac{1}{2}\right) - \frac{1}{4} I$$

$$I = -\frac{4}{5} \frac{1}{2} \frac{1}{2}\left(-e^{\pi} + 1\right) = \frac{e^{\pi} - 1}{5}$$

(h)

$$\int \frac{x^4}{x^3 - 2x^2 + x} dx$$

Hint: first do long division, then use partial fraction decomposition.

2pt

$$\begin{array}{r|l} & x+2 \\ \hline x^3-2x^2+x & x^4 \\ & -\;(x^4-2x^3+x^2) \\ \hline & 2x^3-x^2 \\ & -\;(2x^3-4x^2+2x) \\ \hline & 3x^2-2x \end{array}$$

$$\frac{x^4}{x^3-2x^2+x} = x+2+\frac{3x^2-2x}{x^3-2x^2+x} = x+2+\frac{3x-2}{x^2-2x+1}$$

2pt $\quad = x+2+\dfrac{3x-2}{(x-1)^2}$

2pt

$$\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Want: $3x-2 = A(x-1)+B$

Thus $A=3$, $B=1$

$$\int \frac{x^4}{x^3-2x^2+x} dx = \int \left(x+2+\frac{3}{x-1}+\frac{1}{(x-1)^2} \right) dx$$

$$= \frac{x^2}{2} + 2x + 3\ln|x-1| + \int \frac{1}{(x-1)^2} dx$$

2pt

Put $u = x-1$ ------ $du = dx$

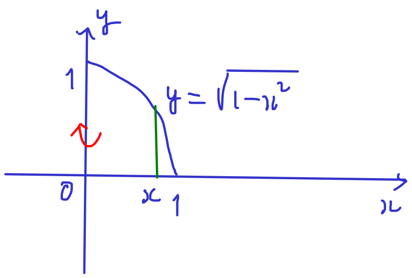
$$\int \frac{1}{(x-1)^2} dx = \int \frac{1}{u^2} du = -u^{-1} + C = \frac{-1}{x-1} + C$$

Conclusion:

$$\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$$

2. Compute the volume of the unit sphere (i.e. sphere with radius 1). Hint: the sphere is a solid of revolution. You can use either the slicing method or shell method.

3pt



First, let's compute the volume of the upper hemisphere:

$$\text{vol} = \int_0^1 2\pi x\sqrt{1-x^2}\,dx \quad \text{(shell method)}$$

$u = 1-x^2$

$du = -2x\,dx$

x	0	1
u	1	0

3pt

$$\int_0^1 2\pi x\sqrt{1-x^2}\,dx = \int_1^0 2\pi\sqrt{u}\left(-\frac{1}{2}\right)du = \int_0^1 \pi\sqrt{u}\,du$$

$$= \frac{2}{3}\pi\, u^{3/2}\Big|_0^1 = \frac{2\pi}{3}$$

The volume of the entire sphere is

$$2\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3}$$