Project B

Due March 6, 2019

I Instruction.

The purpose of this project is to apply integrals to

- Divide a piece of land into 3 pieces of equal areas,
- Compute the volume of a spinning top,
- Determine how fast a sum grows to infinity.

Section II is only for practice. Please do not include it in your report. Section III (the three problems) is what you write a report on. The recommended order to work on this project is:

Problem 1 → Problem 2 → Practice → Problem 3.

If you use Mathematica for any part of the report, please write the (main) commands that you use. You don't need to write all commands. **The report should be written coherently.** You should explain the methods/procedures that you use. You can use only results/methods you already learned from the course (or previous courses). Writing only Mathematica commands without any explanation is not sufficient. Answers that are not supported by valid arguments will receive little or no credits. **You are strongly encouraged to work with your group members.**

II Practice.

For each $n \geq 1$, denote $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$. In other words, S_n is the sum of the square of the first n positive integers. For example, $S_1 = 1$, $S_2 = 5$, $S_3 = 14$, $S_4 = 30$, etc. It is easy to see that $S_n \to \infty$ as $n \to \infty$. The question is how fast S_n goes to infinity. The logarithm function $\ln n$ grows more slowly than n, which grows more slowly than n^2, which grows more slowly than exponential function e^n, which grows more slowly than the factorial function $n!$, etc. To know how fast a sequence goes to infinity is important in many applications. If one knows the formula:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, \tag{1}$$

the answer would be simple: S_n goes to the infinity at the rate of n^3. More specifically,

$$\lim_{n\to\infty} \frac{S_n}{n^3} = \frac{1}{3}. \tag{2}$$

However, such a nice formula as (1) is rare. How can one still achieve (2) without using the identity (1)? Definite integral provides a means to do so. First, we write S_n using Σ notation:

$$S_n = \sum_{k=1}^{n} k^2.$$

The goal is to relate S_n to the Riemann sum of a suitable function. Now divide S_n by n^3 in a somewhat clever way:

$$\frac{S_n}{n^3} = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \frac{k^2}{n^3} = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}$$

This is the Riemann sum of function $f(x) = x^2$ on the interval $[0, 1]$ with grid-points:

$$x_0 = \frac{0}{n},\ x_1 = \frac{1}{n},\ x_2 = \frac{2}{n}, \ldots, x_n = \frac{n}{n}$$

and sample points: $x_k^* = x_k$, which is the right endpoint of the k'th subinterval. The limit of this Riemann sum as $n \to \infty$ is the definite integral of f on $[0,1]$. Therefore,

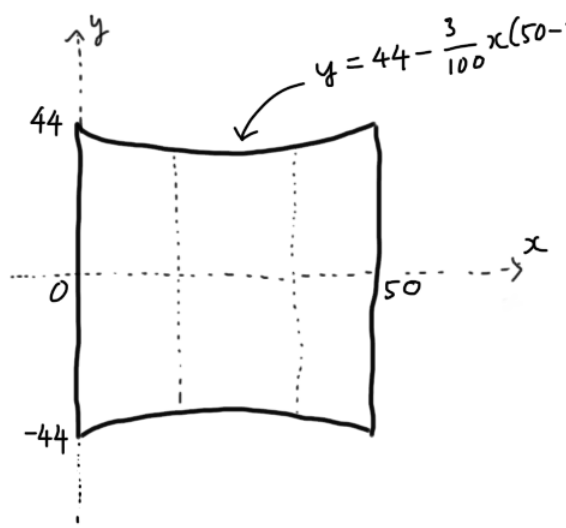
$$\lim_{n\to\infty} \frac{S_n}{n^3} = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

III Problems.

1. There is a man who owns a piece of land shaped as in the picture. It is symmetric with respect to the x-axis. The length of the property is 88 feet, extending from $y = -44$ to $y = 44$. The width is 50 feet. The boundaries along the length are straight lines. The boundaries along the width are parabolae. The upper parabola is given by

$$y = 44 - \frac{3}{100}x(50 - x)$$

The man is old and has 3 children. Before he dies, he wants to divide the property equally into 3 pieces (i.e. with equal areas). Can you suggest how he should divide it?



2. (Similar to Prob. 62, page 432)
Consider a wooden spinning top whose shape is a solid formed when the region bounded by the function

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \le x \le 1, \\ \frac{1}{4}x + \frac{1}{4} & \text{if } 1 < x \le 3, \\ -\frac{1}{4}x + \frac{7}{4} & \text{if } 3 < x \le 4, \end{cases}$$

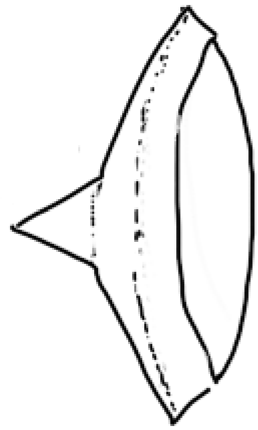
the x-axis and the line $x = 4$ is revolved about the x-axis. See picture.

(a) Graph the function f.

(b) Find the volume of the spinning top.

3. For each $n \ge 1$, denote

$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$$



In other words, S_n is the sum of the reciprocal of the square root of the first n positive integers.

(a) Use Mathematica to evaluate S_n with $n = 5, 10, 50, 200, 1000, 10000$. Does S_n seem to go to infinity as $n \to \infty$?

(b) Use Mathematica to evaluate the quotient $\frac{S_n}{\sqrt{n}}$ for $n = 5, 10, 50, 200, 1000, 10000$. What is your guess for $\lim_{n\to\infty} \frac{S_n}{\sqrt{n}}$?

(c) Use a similar procedure as in the Practice to find $\lim_{n\to\infty} \frac{S_n}{\sqrt{n}}$. Hint: write $\frac{1}{\sqrt{k}\sqrt{n}} = \frac{1}{n}\sqrt{\frac{n}{k}}$.

(d) What can you say about how fast S_n goes to infinity as n goes to infinity?