Solution to selected problems in HW1:

2) $c = (1,-8)$, $a = (1,2)$, $b = (2,-1)$

We need to find x and y such that

$$(1,-8) = \underbrace{x(1,2) + y(2,-1)}$$

$$(x, 2x) + (2y, -y) = (x+2y, 2x-y)$$

We obtain two equations to solve for two unknowns:

$$\begin{aligned} 1 &= x + 2y \\ -8 &= 2x - y \quad \times 2 \end{aligned}$$

Multiply the second eq. by 2, then add to the first eq. (this will eliminate y):

$$-15 = x + 4x = 5x \longrightarrow x = -3$$

Substitute $x=-3$ into the first eq. to find y : $y = 2$

3) a. $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x,y) = (x, 2y, x-y)$$

* Check if f is additive:

Take two vectors in $\mathbb{R}^2$, say

$$v_1 = (x_1, y_1)$$

$$v_2 = (x_2, y_2)$$

- Add v_1 to v_2 : $v_1 + v_2 = (x_1+x_2, y_1+y_2)$

then apply f : $f(x_1+x_2, y_1+y_2) = \boxed{(x_1+x_2, 2(y_1+y_2), x_1+x_2-(y_1+y_2))}$

use the def. of f

- Apply f to v_1 and v_2:

$$f(x_1,y_1) = (x_1, 2y_1, x_1 - y_1)$$

$$f(x_2,y_2) = (x_2, 2y_2, x_2 - y_2)$$

then add : $f(x_1,y_1) + f(x_2,y_2) = (x_1, 2y_1, x_1-y_1) + (x_2, 2y_2, x_2-y_2)$

$$= \boxed{(x_1+x_2, 2y_1+2y_2, x_1-y_1+x_2-y_2)}$$

Two results are the same! f is additive.

* Check if f is scalar multiplicative:

Take any vector $v = (x, y)$,

any number $c \in \mathbb{R}$

- scale first: $cv = c(x, y) = (cx, cy)$

  then apply f: $f(cv) = f(cx, cy) = \boxed{(cx, 2cy, cx - cy)}$

- apply f first: $f(v) = f(x, y) = (x, 2y, x - y)$

  then scale: $cf(v) = c(x, 2y, x-y) = (cx, c2y, c(x-y))$

  $= \boxed{(cx, c2y, cx - cy)}$

Two results are the same! f is scalar multiplicative.

Therefore, f is a linear map.

3 b) $f(x, y) = (y, xy)$

Because of the term xy, we guess that f is not linear.

To prove f is not linear, it suffices to point out that f is either not additive or scalar multiplicative.

Say, let's try to show that f is not scalar multiplicative.

We will point out a counter example:

Pick a vector $v = (1, 1)$ and a scalar factor $c = 4$.

- Scale first: $cv = 4(1, 1) = (4, 4)$

  then apply f: $f(4, 4) = \boxed{(4, 16)}$

- Apply f first: $f(v) = f(1, 1) = (1, 1)$

  then scale: $4(1, 1) = \boxed{(4, 4)}$

Two results are different. Thus, f is not scalar multiplicative.

Conclusion: f is not linear.