Homework 1

1. Consider three following vectors in $\mathbb{R}^2$:

$$a = (1,\, 2), \quad b = (2,\, -1), \quad c = (1,\, -8).$$

   (i) Draw these vectors on the plane.

   (ii) Compute $a + 2b$, $2(a - b) + 3(b + a)$, $|a + b + c|$.

2. With vectors a, b, c given in Problem 1, find the numbers x and y such that $c = xa + yb$.

3. Check if each following map is a linear map. If it is, explain why (by verifying the 2 criteria). If it is not, show how one of these criteria is violated.

   (a) $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x,\, 2y,\, x - y)$.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (y,\, xy)$.

   (c) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = y + z + 1$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + y,\, y)$. Consider two points $A(1, 2)$ and $B(2, 3)$ on the plane.

   (a) Verify that f is a linear map.

   (b) Draw vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC} = 2\overrightarrow{OA} - \overrightarrow{OB}$, and $\overrightarrow{OD} = 3\overrightarrow{OA} - 2\overrightarrow{OB}$ on the plane. Based on the picture, are the points A, B, C, D colinear?

   (c) Compute $f(A)$, $f(B)$, $f(C)$, $f(D)$ and draw these points on the plane. Are they colinear?

   (d) What is a property of linear maps one might deduce from this observation?

5. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}.$$

   Compute the following matrices (if they are well-defined):

   (a) $2A$, $3A - 2B$, C^T.

   (b) A^2, B^2, C^2, $C^T C$.

   (c) AC, CA, AB.

   (d) Check if $AB = BA$.

   (e) Check if $(A + B)C = AC + BC$.

   (f) Check if $(AB)C = A(BC)$.

   (g) Check if $(AC)^T = A^T C^T$.

   (h) Check if $(AC)^T = C^T A^T$.