Solution to selected problems in HW 2

4) The linear map $f: \mathbb{R}^2 \to \mathbb{R}$ is represented by matrix of size 1×2

$$A = [a \;\; b]$$

$$f(x) = Ax$$

Thus,

$$1 = f(1,2) = [a \;\; b]\begin{bmatrix}1\\2\end{bmatrix} = a + 2b$$

$$4 = f(2,5) = [a \;\; b]\begin{bmatrix}2\\5\end{bmatrix} = 2a + 5b$$

We get a system $\begin{cases} a + 2b = 1 \\ 2a + 5b = 4 \end{cases}$

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 5 & 4 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\begin{cases} b = 2 \\ a = 1 - 2b = -3 \end{cases}$$

Therefore, $A = [-3 \;\; 2]$ and

$$f(x_1, x_2) = A\begin{bmatrix}x_1\\x_2\end{bmatrix} = [-3 \;\; 2]\begin{bmatrix}x_1\\x_2\end{bmatrix} = -3x_1 + 2x_2$$

5) Matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 \\ 1 & -3 & 4 & 0 \\ -1 & -4 & 3 & -2 \end{bmatrix}$$

is of size 3×4. It represents a linear map $f: \mathbb{R}^4 \to \mathbb{R}^3$.

Domain : $\mathbb{R}^4$

Target set : $\mathbb{R}^3$

Explicit formula : $f(x) = Ax$, which is

$$f\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = \begin{bmatrix} 2 & 0 & 1 & 3 \\ 1 & -3 & 4 & 0 \\ -1 & -4 & 3 & -2 \end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_3 + 3x_4 \\ x_1 - 3x_2 + 4x_3 \\ -x_1 - 4x_2 + 3x_3 - 2x_4 \end{bmatrix}$$

In vector form:

$$f(x_1, x_2, x_3, x_4) = (2x_1 + x_3 + 3x_4,\ x_1 - 3x_2 + 4x_3,\ -x_1 - 4x_2 + 3x_3 - 2x_4).$$

7)

$$f(x_1, x_2) = (2x_2, x_1)$$

$$g(x_1, x_2) = (x_2, x_1 - x_2, x_1)$$

$$h(x_1, x_2) = (0, 3x_2 - 2x_1)$$

f is represented by matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

g " $B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$

h " $C = \begin{bmatrix} 0 & 0 \\ -2 & 3 \end{bmatrix}$

(i)

$f + h$ is a map such that

$$(f+h)(x_1, x_2) = f(x_1, x_2) + h(x_1, x_2) = (2x_2, 3x_2 - x_1)$$

It is represented by matrix $\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ (which is $A + C$)

(iii)

$g \circ f$ is the map such that $g \circ f(x_1, x_2) = g(f(x_1, x_2))$

It is represented by matrix

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 2 \end{bmatrix}$$ (call it E)

Explicit formula for $g \circ f$:

$$g\circ f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = E\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1+2x_2 \\ 2x_2 \end{bmatrix}$$

Hence, $g\circ f(x_1, x_2) = (x_1, -x_1+2x_2, 2x_2)$

8. (Problem 7 on page 53 in the textbook)

$$\begin{cases} x + 2y - z = 1 \\ 2x + y + z = 1 \\ x - y + 2z = 1 \end{cases}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \text{(echelon form)}$$

The last row : $0 = 1$

The system is inconsistent. There are no solutions.