Solutions to some problems of HW3:

1e)
$$\begin{cases} x_1 - 8x_3 + 7x_4 = 0 \\ x_1 + x_2 - 2x_3 + 2x_4 = 0 \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 = 0 \end{cases}$$

Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & -8 & 7 & 0 \\ 1 & 1 & -2 & 2 & 0 \\ 4 & 5 & -2 & 3 & 0 \end{array}\right] \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & -8 & 7 & 0 \\ 0 & 1 & 6 & -5 & 0 \\ 0 & 5 & 30 & -25 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 5R_2} \left[\begin{array}{cccc|c} 1 & 0 & -8 & 7 & 0 \\ 0 & 1 & 6 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ This is REF.

these cols. don't have pivot entries

The system has infinitely many solutions, with x_3 and x_4 being free variables.

$$x_3 = s$$
$$x_4 = t$$

From the second row: $x_2 + 6x_3 - 5x_4 = 0$

$$\Longrightarrow x_2 = -6x_3 + 5x_4 = -6s + 5t$$

From the first row:

$$x_1 - 8x_3 + 7x_4 = 0$$
$$\Longrightarrow x_1 = 8x_3 - 7x_4 = 8s - 7t$$

Conclusion:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8s - 7t \\ -6s + 5t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 8 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

where s and t are arbitrary numbers.

3) $f(x) = (-2, 12, -20)$

In matrix form:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ -3 & -8 \\ 1 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \begin{bmatrix} -2 \\ 12 \\ -20 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ -3 & -8 & 12 \\ 1 & 8 & -20 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + 3R_1} \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -2 & 6 \\ 0 & 6 & -18 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + 3R_2} \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{array}\right]$$ This is in REF

Every column has a pivot entry. The system, therefore, has a unique solution.

From second row: $-2x_2 = 6 \Rightarrow x_2 = -3$

From first row: $x_1 + 2x_2 = -2 \Rightarrow x_1 = -2 - 2(-3) = 4$

Therefore,

$$x = (x_1, x_2) = (4, -3).$$

4b) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{bmatrix}$

There are many ways to turn this matrix into reduced row echelon form (RREF). One way is: first turn it into REF, then use each pivot entry to turn the column containing it into pivot column.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{bmatrix} \xrightarrow[R_3 = R_3 + 2R_1]{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 3 & -4 \end{bmatrix} \xrightarrow{R_3 = R_3 + 3R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$ this is in REF

Now turn each col. that contains a pivot entry (in this case all three columns) into a pivot column.

How? First turn each pivot entry to 1.

$$\xrightarrow[R_1 = R_1 - 2R_2]{R_2 = R_2(-1)} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow[\substack{R_1 = R_1 - 3R_3 \\ R_2 = R_2 + 2R_3}]{R_3 = R_3/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ this is in RREF

5 d)

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 + \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & 1/2 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$ this is in REF

this col. has no pivot entries

Conclusion: the given matrix is not invertible.

6)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 5 & 4 \\ 13 & 10 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ -4 & 5 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

this matrix represents f^{-1}. Explicit formula for f^{-1}:

$$f^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ -4 & 5 & -2 \\ 5 & -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ -4x_1 + 5x_2 - 2x_3 \\ 5x_1 - 3x_2 + x_3 \end{bmatrix}$$

or in vector form:

$$f^{-1}(x_1, x_2, x_3) = (-2x_2 + x_3, -4x_1 + 5x_2 - 2x_3, 5x_1 - 3x_2 + x_3).$$