Solutions to some problems in HW4

5) (a) $A - \lambda I_2 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix}$

This matrix fails to be invertible if its determinant is equal to 0.

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5$$

Set $\lambda^2 - 6\lambda + 5 = 0$. This quadratic polynomial has two roots

$$\lambda_1 = 1 \text{ and } \lambda_2 = 5$$

Conclusion: for $\lambda = 1$ or 5, matrix $A - \lambda I_2$ fails to be invertible.

(b) $Av = 5v = 5I_2 v$

$\Rightarrow (A - 5I_2)v = 0$

$$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 + R_1} \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free variable

Pick $x_2 = 3$. The first equation: $-3x_1 + x_2 = 0$, thus $x_1 = 1$.

$\boxed{v = (1, 3)}$

* Note: Each choice of x_2 gives a vector v. There are infinitely many vectors v satisfying $Av = 5v$. These vectors differ from one another only by a scaling factor.

6) b. $\begin{vmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{vmatrix} = \begin{vmatrix} -1 & 3 & 2 \\ 3 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} \begin{matrix} -1 & 3 \\ 3 & 1 \\ -2 & 0 \end{matrix} = -1 + 6 + 0 - (-4 + 0 + 9) = 0$

v_1, v_2, v_3 are LD.

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 3 & 1 & -1 & 0 \\ -2 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + 3R_1} \left[\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 0 & 10 & 5 & 0 \\ 0 & -6 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 + 6R_2]{R_2 = R_2/5} \left[\begin{array}{ccc|c} 4 & 3 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

this column has no pivot entries

c_3 is a free variable.

Pick $c_3 = 2$.

-rom the second row: $2c_2 + c_3 = 0 \Rightarrow c_2 = -1$

From the first row: $-c_1 + 3c_2 + 2c_3 = 0 \Rightarrow c_1 = 3c_2 + 2c_3 = 1$

Therefore,

$$\boxed{v_1 - v_2 + 2v_3 = 0}$$

d. $\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 2 \\ 0 & 1 & -2 \\ 1 & 0 & 4 \end{bmatrix} \xrightarrow[R_4 = R_4 - R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

this col. has no pivot entries

v_1, v_2, v_3 are LD.

c_3 is free variable.

Pick $c_3 = 1$. The from the second row: $c_2 - 2c_3 = 0$

$\Rightarrow c_2 = 2$

From the first row: $c_1 + 4c_3 = 0 \Rightarrow c_1 = -4$

Therefore,

$$\boxed{-4v_1 + 2v_2 + v_3 = 0}$$