Homework 4

1. This problem is an application of inverse matrix: *If a system of linear equations has the same number of equations as unknowns and if the coefficient matrix is invertible, then the system can be solved by inverse matrix.* Consider the system:

$$\begin{cases} 2y - z &= -2 \\ 5x + 2y + 3z &= 4 \\ 7x + 3y + 4z &= -5 \end{cases}$$

   (i) Write the system in matrix form $AX = b$.

   (ii) Find $\det A$. Is A invertible?

   (iii) Find A^{-1}.

   (iv) Use the formula $X = A^{-1}\mathrm{b}$ to find solutions to the system.

2. Check if each of the following statements is true for all 2-by-2 matrices A and B. If it is, justify your answer. If it is not, give a counterexample.

   (a) $(A + B)^{-1} = A^{-1} + B^{-1}$ (assuming A and B are invertible)

   (b) $(AB)^{-1} = B^{-1}A^{-1}$ (assuming A and B are invertible)

   (c) $AA^T = A^TA$ (Recall that A^T denotes the transpose matrix.)

3. Find the determinant of the following matrices.

   (a)
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   (b)
$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

   (c)
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 4 & -5 \end{bmatrix}$$

   (d)
$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$$

4. Find all values c such that the following matrix is invertible

$$\begin{bmatrix} 1 & c & 0 \\ c & 1 & 0 \\ 0 & 1 & c \end{bmatrix}$$

5. Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

   (a) Determine all numbers λ such that the matrix $A - \lambda I_2$ fails to be invertible.

   (b) Find a nonzero vector v such that $Av = 5v$. (Hint: write $5v$ as $5I_2v$.)

6. (Problems 2, 5, 6, 8 of Section 5.4 on page 76 textbook) Determine if the given vectors are linearly dependent or linearly independent. If they are linearly dependent, find a nontrivial linear combination of the vectors that has sum 0.

(a) $v_1 = (3,\, 2)$, $v_2 = (6,\, 4)$.

(b) $v_1 = (-1,\, 3,\, -2)$, $v_2 = (3,\, 1,\, 0)$, $v_3 = (2,\, -1,\, 1)$.

(c) $v_1 = (2,\, 1,\, 0)$, $v_2 = (0,\, 1,\, 0)$, $v_3 = (-1,\, 2,\, 0)$.

(d) $v_1 = (1,\, 1,\, 0,\, 1)$, $v_2 = (0,\, 1,\, 1,\, 0)$, $v_3 = (4,\, 2,\, -2,\, 4)$.

7. Determine (i.e write an explicit formula for) a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(2,\, 1) = (-1, 3)$ and $f(3,\, 2) = (2, 0)$. (Hint: find the matrix associated with f.)

8. Let L be the line given by the equation $3x - 4y = 0$ on the plane. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the (orthogonal) projection onto L. That is, f maps each point on the plane to its projection on L. Recall that a point can be thought as a vector based at the origin, and vice versa. One can write a formula for f by following the below steps:

   (i) Draw the line L. (Can you convince yourself using picture that f should be a linear map? For example, draw any two vectors on the plane. Is the projection of the sum equal to the sum of the projections? Similar question for scaling.)

   (ii) Find a unit direction vector of L (i.e. a vector that has length 1 and is parallel to L).

   (iii) Determine the projections of vectors $e_1 = (1,\, 0)$ and $e_2 = (0,\, 1)$ onto L. (Hint: use dot product.)

   (iv) What are $f(e_1)$ and $f(e_2)$? Determine the matrix A associated with f.

   (v) Write an explicit formula for f (i.e. $f(x_1,\, x_2) = \ldots$)