<u>Solution to Prob 1, HW 6</u>

$$f(x) = e^{x/2} \sin\left(\frac{x}{2}\right)$$

By the product rule,

$$f'(x) = \frac{1}{2} e^{x/2} \sin\left(\frac{x}{2}\right) + e^{x/2}\, \frac{1}{2} \cos\left(\frac{x}{2}\right) = e^{x/2}\left(\frac{1}{2}\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}\right)$$

$$f''(x) = \frac{1}{2} e^{x/2}\left(\frac{1}{2}\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}\right) + e^{x/2}\left(\frac{1}{4}\cos\frac{x}{2} - \frac{1}{4}\sin\frac{x}{2}\right)$$

$$= e^{x/2}\left(\cancel{\frac{1}{4}\sin\frac{x}{2}} + \frac{1}{4}\cos\frac{x}{2} + \frac{1}{4}\cos\frac{x}{2} - \cancel{\frac{1}{4}\sin\frac{x}{2}}\right)$$

$$= \frac{1}{2} e^{x/2} \cos\frac{x}{2}$$

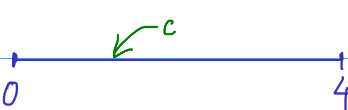
We see that

$$|f'(x)| \le e^{x/2}\left(\frac{1}{2}\underbrace{\left|\sin\frac{x}{2}\right|}_{\le 1} + \frac{1}{2}\underbrace{\left|\cos\frac{x}{2}\right|}_{\le 1}\right) \le e^{x/2}$$

$$|f''(x)| \le \frac{1}{2} e^{x/2}\left|\cos\frac{x}{2}\right| \le \frac{1}{2} e^{x/2} \le e^{x/2}$$

Lagrange's theorem says that there exists c between 0 and 4 such that

$$f(4) - T_n(4) = R_n(4) = \frac{f^{(n+1)}(c)}{(n+1)!} \underbrace{(4-0)^{n+1}}_{4^{n+1}}$$



Estimate $R_n(4)$:

$$|R_n(4)| \le \frac{|f^{(n+1)}(c)|}{(n+1)!} 4^{n+1}$$

$$\le \frac{e^{c/2}}{(n+1)!} 4^{n+1} \le \frac{e^2}{(n+1)!} 4^{n+1} \le \frac{8(4^{n+1})}{(n+1)!}$$

To guarantee that $|R_n(4)| < 10^{-2}$, one only needs n such that

$$\frac{8\,(4^{n+1})}{(n+1)!} < 10^{-2}$$

By trying $n = 1, 2, 3, \ldots$ we see that $n = 14$ will do it.