Homework 6

1. This exercise is an example of how find the lowest order of Taylor polynomial to compute $e^2 \sin 2$ with error of at most 10^{-2}. Put

$$f(x) = e^{\frac{x}{2}} \sin\left(\frac{x}{2}\right)$$

   (a) Compute $f'(x)$ and $f''(x)$.
   (b) Find the Taylor polynomials T_0, T_1, T_2 of f about the base point 0.
   (c) From Part (a), verify that $|f'(x)| \leq e^{x/2}$ and $|f''(x)| \leq e^{x/2}$.
   (d) What does Lagrange's theorem say about the error term $R_n(4) = f(4) - T_n(4)$?
   (e) It is know that (you don't have to verify) $|f^{(n)}(x)| \leq e^{x/2}$ for any $x \in \mathbb{R}$. Find an upper bound in terms of n for the error term $R_n(4)$. (Hint: use $e^2 < 8$).
   (f) With the upper bound found in Part (e), use your calculator to find n (by trying $n = 1, 2, 3, \ldots$) such that $|R_n(4)| < 10^{-2}$.

2. Consider the series

$$1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \frac{1}{3^5} + \ldots$$

   (a) Express this series using Σ notation.
   (b) Let S_n be the n'th partial sum, which is the sum of the first n terms. Find a general formula of S_n in terms of n. Hint: write $(-1/3)S_n$, then subtract S_n from it.
   (c) Find $\lim_{n\to\infty} S_n$.
   (d) Does the series converge? If it does, what is the sum of the series?

3. Consider the series

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots$$

   (a) Express this series using Σ notation.
   (b) Verify that

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}}$$

   (c) Based on the above formula, verify that for every $k \geq 1$

$$\sqrt{k+1} - \sqrt{k} < \frac{1}{2\sqrt{k}}$$

   (d) Let S_n be the n'th partial sum, which is the sum of the first n terms. Verify that $S_n > 2(\sqrt{n+1} - 1)$.
   (e) Find $\lim_{n\to\infty} S_n$.
   (f) Does the series converge or diverge?