Some review problems for Midterm

1. Compute the eigenvalues of the following matrices. To each eigenvalue, find the corresponding eigenvectors . Pick only the eigenvectors that are linearly independent.

   (a) $$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

   (b) $$\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$

   (c) $$\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$$

   (d) $$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

2. Determine whether the given vectors are linearly independent. If they are linearly dependent, find a nontrivial linear combination of the vectors that has sum 0.

   (a) $v_1 = (0, 0)$, $v_2 = (2, -1)$.
   (b) $v_1 = (1, 2)$, $v_2 = (2, 0)$, $v_3 = (3, 1)$.
   (c) $v_1 = (1, 3, -1)$, $v_2 = (3, 7, -7)$, $v_3 = (1, 2, -3)$.
   (d) $v_1 = (0, 0, 1, 1)$, $v_2 = (1, 1, 0, 0)$, $v_3 = (1, 0, 0, -1)$.

3. Determine whether the each of the following matrices is invertible. If it is, find the inverse matrix A^{-1}.

   (a) $$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

   (b) $$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

   (c) $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

   (d) $$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{bmatrix}$$

4. Let A, B, C be 2-by-2 matrices such that $\det A = 2$, $\det B = 3$, $\det C = 1$. Find the determinant of matrix $D = -A^2B^{-2}C^{-1}$.

5. Solve the following systems of linear equations using Gauss elimination method. If a system has infinitely many solutions, write them in parametric vector form.

   (a) $$\begin{cases} 3x + 4y = 10 \\ 2x + 3y = 7 \end{cases}$$

   (b) $$\begin{cases} x + y + z = 6 \\ 2y + 5z = -4 \\ 2x + 5y - z = 27 \end{cases}$$

   (c) $$\begin{cases} x + y + 2z = 4 \\ x - 2y + z = 0 \\ x - 5y = -4 \end{cases}$$

   (d) $$\begin{cases} x + 2y + 6z = 5 \\ -x + y - 2z = 3 \\ x - 4y - 2z = 1 \end{cases}$$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that $f(-1, 2) = (0, 2, 1)$ and $f(-2, 3) = (-1, 3, 0)$. Find an explicit formula of f. That is, a formula of the form $f(x, y) = (ax + by, cx + dy)$ where a, b, c, d are constants.

7. Determine whether the following maps are linear. Explain why or why not.

   (a) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (2x_1, x_1 + x_2)$.

   (b) $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, \sin x)$.

8. Each of the following maps is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Do the following:

   (i) Write the matrix associated with it.

   (ii) Give a geometric description of the map.

   (a) $f(x, y) = (0, y)$

   (b) $f(x, y) = (x, -y)$

   (c) $f(x, y) = (2x, 2y)$,

   (d) $f(x, y) = (y, x)$,

   (e) $f(x, y) = (-y, x)$,

   (f) $f(x, y) = (-3y, 3x)$

9. Is there any 2-by-2 matrix A with real (not complex) entries such that $A^2 = -I_2$? If yes, give an example. If no, explain why.