MATH 306, SECTION 20, MIDTERM EXAM, WINTER 2019

Name	Student ID

- Read the instruction of each problem carefully.
- The exam has 6 pages. **Circle your final results.**
- To get full credit for a problem **you must show your work**. Answers not supported by valid arguments will get little or no credit.

Problem	Possible points	Earned points
1	10	
2	15	
3	15	
4	15	
5	15	
6	15	
Total	85	

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Problem 1. (10 pts) Determine if the following map is linear. Explain why and why not.

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = (2x,\, 1 - x)$$

$$\left.\begin{array}{l} 2f(1) = 2(2,0) = (4,0) \\ f(2) = (4,-1) \end{array}\right\} f(2) \neq 2f(1)$$

f is not linear because it is not scaling multiplicative.

Problem 2. Consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x, y) = (x + 3y,\, 2x + 4y)$$

(a) (5 pts) Find the matrix A associated with f.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

(b) (5 pts) Determine if A is invertible. If it is, find A^{-1}.

$\det A = 1(4) - 2(3) = -2 \neq 0$

A is invertible

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

(c) (5 pts) Determine if f is invertible. If it is, find an explicit expression for f^{-1}. That is, a formula of the form $f^{-1}(x, y) = (ax + by,\ cx + dy)$ where a, b, c, d are constants.

$$f^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x + \frac{3}{2}y \\ x - \frac{y}{2} \end{bmatrix}$$

$$f^{-1}(x, y) = \left(-2x + \frac{3}{2}y,\ x - \frac{1}{2}y\right)$$

Problem 3. (15 pts) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map associated with matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{bmatrix}$$

Solve the equation $f(x) = (1, 3, -3)$. That is, find all vectors $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $f(x_1, x_2, x_3) = (1, 3, -3)$. If there are infinitely many solutions, write them in parametric vector form.

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -2 & 1 & 8 & 3 \\ 1 & -2 & -7 & -3 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 5 & 10 & 5 \\ 0 & -4 & -8 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 + 4R_2]{R_2 = R_2/5} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$ (free variable)

The system has infinitely many solutions.

From 2^{nd} row : $x_2 + 2x_3 = 1$. Thus, $x_2 = 1-2t$

From 1^{st} row : $x_1 + 2x_2 + x_3 = 1$. Then $x_1 = 1 - 2(1-2t) - t = -1 + 3t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1+3t \\ 1-2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Problem 4. (15 pts) Determine if the following vectors are linear dependent or independent:

$$\begin{aligned} v_1 &= (2,\, 1,\, 0) \\ v_2 &= (2,\, -1,\, 2) \\ v_3 &= (0,\, -1,\, 1) \end{aligned}$$

If they are linearly dependent, find a nontrivial linear combination of the vectors that has sum 0.

$$\left[\begin{array}{ccc|c} | & | & | & | \\ v_1 & v_2 & v_3 & 0 \\ | & | & | & | \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \Leftrightarrow R_3]{R_1 = R_1/2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

c_3 is free variable.

Pick $c_3 = 2$.

From the 2^{nd} row: $2c_2 + c_3 = 0$. Then $c_2 = -1$

From the 1^{st} row: $c_1 + c_2 = 0$. Then $c_1 = 1$

v_1, v_2, v_3 are linearly dependent, and

$$v_1 - v_2 + 2v_3 = 0.$$

Problem 5. (15 pts) Is there any linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(3, 2) = (-1, 2, 0)$ and $f(2, 1) = (2, 0, -2)$? If yes, write the matrix associated with f. If no, explain why.

$v_1 = (3,2)$

$v_2 = (2,1)$

A .... matrix associated with f.

$$\left.\begin{array}{l} f(v_1) = A\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix} \\ f(v_2) = A\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix} \end{array}\right\} \quad A\underbrace{\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}} = \begin{bmatrix}-1 & 2\\2 & 0\\0 & -2\end{bmatrix}$$

invertible matrix because

$\det = 3(1) - 2(2) = -1 \neq 0$

So yes, there is such a map f.

$$A = \begin{bmatrix}-1 & 2\\2 & 0\\0 & -2\end{bmatrix}\underbrace{\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}^{-1}}_{=\frac{1}{-1}\begin{bmatrix}1 & -2\\-2 & 3\end{bmatrix}}$$

$$= \begin{bmatrix}-1 & 2\\2 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}-1 & 2\\2 & -3\end{bmatrix} = \begin{bmatrix}5 & -8\\-2 & 4\\-4 & 6\end{bmatrix}$$

Problem 6. (15 pts) Consider the linear map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (x + 3y,\, 2x + 6y).$$

Find the eigenvalues and eigenvectors of f. (That is, the eigenvalues and eigenvectors of the matrix associated with f.) Pick only the eigenvectors that are linearly independent.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

$$A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = (1-\lambda)(6-\lambda) - 2(3) = \lambda^2 - 7\lambda = \lambda(\lambda - 7)$$

Two roots: $\lambda = 0$ and $\lambda = 7$.

* With $\lambda = 0$:

$$A - 0I = A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

$$[A \mid 0] = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$$x_1 + 3x_2 = 0 \implies x_1 = -3t$$

Eigenvectors: $(-3t, t) = t(-3, 1)$. $v_1 = (-3, 1)$

* With $\lambda = 7$:

$$[A - 7I_2 \mid 0] = \left[\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow[R_2 = R_2 + R_1]{R_1 = \frac{1}{3}R_1} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$$-2x_1 + x_2 = 0$$

$$\Rightarrow x_1 = t/2$$

Eigenvectors: $\left(\frac{t}{2}, t\right) = t\left(\frac{1}{2}, 1\right)$

$v_2 = \left(\frac{1}{2}, 1\right)$