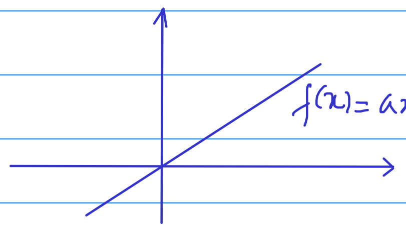
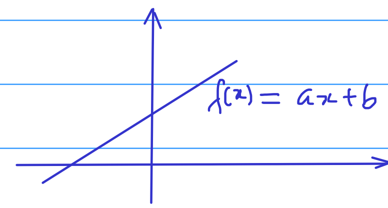
Lecture 1 (1/7/2019)

How to compute the values of a map?

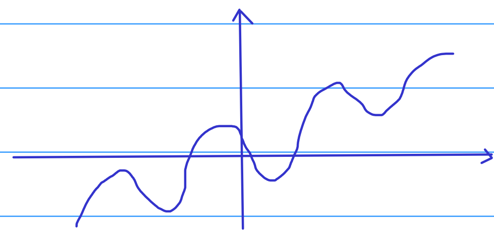
Given a map $f: \mathbb{R} \to \mathbb{R}$, one can describe f by its graph.



linear map



affine map



nonlinear map

Linear map: map whose graph is a line passing through the origin

Linear map is completely determined by number a, the slope of the line.

$f(x) = ax$ .... $\begin{cases} \text{one multiplication} \\ \text{if we know } f(1)\text{, we know } f(x) \text{ for every } x \end{cases}$

$f(x) = ax^2 + bx + c$ .... $\begin{cases} \cdot\ 2+1 = 3 \text{ multiplications} \\ \cdot\ 2 \text{ additions} \\ \cdot \text{ if we know } f(0), f'(0), f''(0)\text{, can determine } a, b, c \text{ and thus } f(x) \end{cases}$

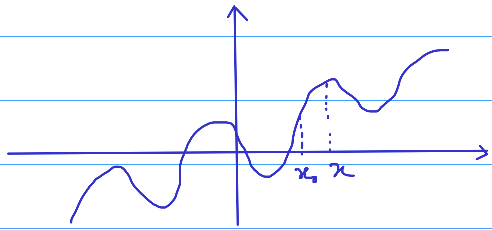
How about $f(x) = \sin x$ or e^x? How can a calculator compute $e^{\sqrt{2}}$ based on additions and multiplications?

We need to approximate. To obtain exact computation, addition and multiplication are not enough. We need calculus.

A nonlinear map is more difficult to calculate.

Suppose one knows everything about f at point x_0. How to compute $f(x)$?

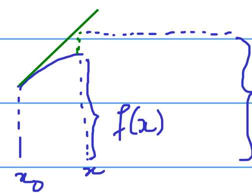
If x is time-variable, this is the problem of predicting the future given all necessary information at the initial time.



If $x \approx x_0$, $f(x) \approx f(x_0)$

If x is further away,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$



If x is further away,

$$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{g(x)} + \frac{f''(x_0)}{2}(x-x_0)^2 \qquad (*)$$

Why?

$$h(x) = f(x) - g(x)$$

Then $h(x_0) = h'(x_0) = 0$. By L'Hospital rule,

$$\lim_{x \to x_0} \frac{h(x)}{(x-x_0)^2} = \lim_{x \to x_0} \frac{h'(x)}{2(x-x_0)} = \lim_{x \to x_0} \frac{h''(x)}{2} = \frac{h''(x_0)}{2}$$

Thus,

$$\frac{h(x)}{(x-x_0)^2} \approx \frac{h''(x_0)}{2}$$ which leads to $(*)$.

For a large class of function f,

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3 + \cdots$$

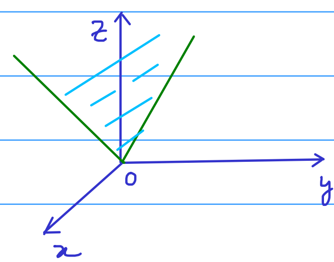
(infinite sum)

We will make rigorous the definition of this sum later in the course. Such a sum is called series (in this case, Taylor series of function f).

Linear map is roughly speaking the 1st order approximation of a nonlinear map. The strength advantage of linear map is more clear in higher dimension.

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(\underbrace{x,y}_{v}) = 2x + 3y \quad \text{linear map}$$

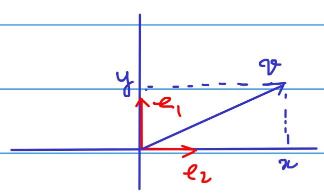


$$f(\overbrace{1,0}^{e_1}) = 2 \qquad f(v) = \underbrace{(2,3)}_{a} \cdot v$$

$$f(\underbrace{0,1}_{e_2}) = 3$$

$$f(v) = x f(e_1) + y f(e_2)$$

Knowing f at e_1 and e_2, one can find $f(v)$ for any vector v.



The calculation of linear maps becomes algebra.

In the first half of the course, we'll study the algebra of linear map.

$$f(v) = \underbrace{A}_{\text{matrix}} v$$

Matrix ~ graph of a map

In the second half, we'll study the calculus of nonlinear map.

Series ~ approximation

* Review on vectors:

$\mathbb{R}^n$ ..... $x = (x_1, x_2, \ldots, x_n)$: vector of n components

$$x = (x_1, x_2, \ldots, x_n)$$
$$y = (y_1, y_2, \ldots, y_n)$$

Sum:

$$x + y = (x_1 + y_1, \ldots, x_n + y_n)$$

Scalar mult.: $(c \in \mathbb{R})$

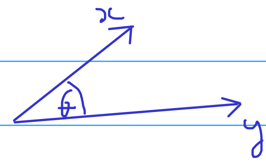
$$cx = (cx_1, cx_2, \ldots, cx_n)$$

Dot product:

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

Length:

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$



$$x \cdot y = |x| |y| \cos\theta$$

Functions on $\mathbb{R}^2$: $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{x+y}$

$f(x,y) = \sin(x^2 + y)$ ....

Their graphs are surfaces in $\mathbb{R}^3$.