Lecture 10 (1/30/2019)

Thm: A square matrix A is invertible if and only if $\det A \neq 0$.
(We will not prove this theorem.)

In this course, we only define determinant of 2×2 and 3×3 matrices.

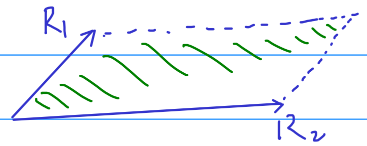
* Important identities:

$$\det I_n = 1$$

$$\det(AB) = \det A \det B \quad \text{(determinant is multiplicative)}$$

This property is not trivial: it's not straightforward from the def. of determinant. However, it can be seen from the geometric interpretation of determinant.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} - R_1 - \\ - R_2 - \end{bmatrix} \qquad \begin{array}{l} R_1 = (a,b) \\ R_2 = (c,d) \end{array}$$



$\det A$ = area of parallelogram formed by vectors R_1 and R_2.

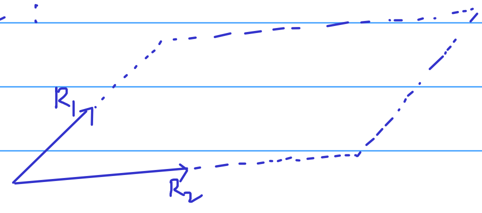
Applications:

- If A is invertible then $\underbrace{\det(AA^{-1})}_{\underbrace{\det I_n}_{1}} = \det A \det A^{-1}$

  Thus, $\det A^{-1} = \dfrac{1}{\det A}$ "det. of inverse is equal to inverse of det."

- Product of invertible matrices is also invertible.
- $\det(\alpha A) = \alpha^n \det A$, where n is the size of A.

<u>Ex</u>: when $A \dots 2\times2$:



$$\det(2A) = 4 \det A$$

This can be seen from the definition of determinant.

Ex:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = ad - bc$$

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}, \quad \det(\alpha A) = (\alpha a)(\alpha d) - (\alpha b)(\alpha c)$$

$$= \alpha^2 (ad - bc)$$

- $A, B, C \ldots$ 3×3 matrices with $\det A = 2$, $\det B = 3$, $\det C = 4$

What is $\det(2A^2BC^{-2})$?

$$\det(2A^2BC^{-2}) = 2^3 \det(A^2BC^{-2}) = 8 \det(A^2) \det B \det(C^{-2})$$

$$= 8 \det(A)^2 \det B (\det C)^{-2}$$

$$= 8 (2)^2\, 3\, (4)^{-2}$$

$$= 6$$

- $\det A$ is a tool for us to check if A is invertible without trying to compute A^{-1}. This is helpful when we are to check if some vectors are linearly independent

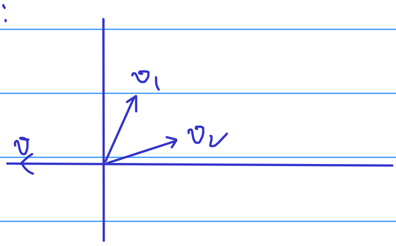
* Linear combinations

A linear combination of vectors $v_1, v_2, \ldots, v_k$ is a vector of the form $v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$.

numbers

v is generated by $v_1, v_2, \ldots, v_k$ through scalings and additions.

Ex:



$v_1 = (1, 2)$

$v_2 = (2, 1)$

Vector $v = (-3, 0)$ is a linear comb. of v_1 and v_2 because $v = v_1 - 2v_2$

Ex: Every vector in $\mathbb{R}^2$ is a linear combination of $e_1 = (1,0)$ and $e_2 = (0,1)$.

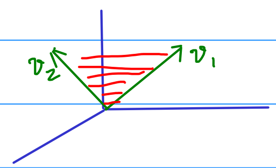
Why? Vector $v = (a,b)$ is equal to $ae_1 + be_2$.

Ex:

e_2 is not a linear comb. of e_1 because one can't find a number c such that $\underbrace{e_2}_{(0,1)} = \underbrace{\alpha e_1}_{(\alpha, 0)}$.

Ex:

Any linear comb. of vector $v_1 = (1,2,3)$ and $v_2 = (2,-1,0)$ must lie on the same plane that contains v_1 and v_2.



Ex: $\underbrace{0}_{\text{vector } 0}$ is a linear comb of any set of vectors because

$$\underbrace{0}_{\text{vector } 0} = \underbrace{0v_1 + 0v_2 + \dots + 0v_k}_{\text{such a linear combination (i.e. with all scaling factors equal to 0) is called a trivial combination.}}$$

number 0