Lecture 12 (2/4/2019)

Applications of linear dependence (LD) and linear independence (LI) of vectors:

Thm: A linear map $f: \mathbb{R}^n \to \mathbb{R}^m$ is uniquely defined if we know f at n LI vectors $v_1, v_2, \dots, v_n$ in $\mathbb{R}^n$.

In other words, if we know $f(v_1), f(v_2), \dots, f(v_n)$, we can compute $f(v)$ for any $v \in \mathbb{R}^n$.

Ex: Find a linear map $f: \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$\left.\begin{aligned} f(1,2) &= (1,2,3) \\ f(2,3) &= (2,-1,2) \end{aligned}\right\} (*)$$

Vectors $v_1 = (1,2)$ and $v_2 = (2,3)$ are LI because neither of them is a multiple of the other. There is only one linear map that satisfies $(*)$. How to find f?

$$f \text{ ------ } A \quad 3\times 2$$

$$f(x) = Ax$$

one needs to find A

$$f(v_1) = (1,2,3) \Rightarrow A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$f(v_2) = (2,-1,2) \Rightarrow A\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2\\-1\\2\end{bmatrix}$$

Combine these two equations into one equation:

$$A\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & -1\\3 & 2\end{bmatrix}$$

Then

$$A = \begin{bmatrix}1 & 2\\2 & -1\\3 & 2\end{bmatrix}\underbrace{\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}^{-1}}_{= \frac{1}{1(3)-2(2)}\begin{bmatrix}3 & -2\\-2 & 1\end{bmatrix} = \begin{bmatrix}-3 & 2\\2 & -1\end{bmatrix}} = \begin{bmatrix}1 & 0\\-8 & 5\\-5 & 4\end{bmatrix}$$

Therefore, $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -8x_1 + 5x_2 \\ -5x_1 + 4x_2 \end{bmatrix}$

Conclusion: $f(x_1, x_2) = (x_1, -8x_1 + 5x_2, -5x_1 + 4x_2)$.

* __Note__: If we know $f(v_1), f(v_2), \dots, f(v_n)$ where $v_1, v_2, \dots, v_n$ are LD then one either lack information to determine f, or (in certain situations) is able to conclude that f doesn't exist.

__Ex__:

Find a linear map $f: \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$f(1,2) = (0,2,-1)$$
$$f(2,4) = (0,4,-2)$$

$v_1 = (1,2)$ and $v_2 = (2,4)$ are LD because $v_2 = 2v_1$.

The second equation can be inferred by the first equation, and this is redundant information. There are in fact infinitely many linear maps from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $f(1,2) = (0,2,-1)$.

Why?

Given a triple (a,b,c), there is a linear map $f: \mathbb{R}^2 \to \mathbb{R}^3$ such that

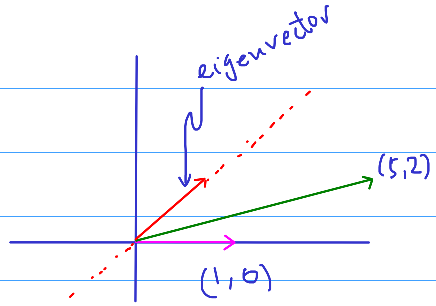
$$f(1,2) = (0,2,-1)$$
$$f(2,3) = (a,b,c)$$

__Ex__ There is no linear map $f: \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$f(1,2) = (0,2,-1)$$
$$f(2,4) = (0,4,-3)$$

why? $v_2 = 2v_1$ but $f(v_2) \neq 2f(v_1)$.

* Eigenvalues and eigenvectors:



$f: \mathbb{R}^2 \to \mathbb{R}^2$ takes vector (x_1, x_2) to $(x_1 + 2x_2,\ 2x_1 + x_2)$.

Each nonzero vector on the plane specifies a direction.

An eigenvector of f is a vector $v \neq 0$ such that $f(v)$ is parallel to v. In other words,

$$f(v) = \lambda v \quad \text{for some scalar } \lambda.$$

λ is called the eigenvalue corresponding to eigenvector v.

* Observation: on the direction of eigenvectors, f acts by scaling (no change of direction).

$$f(1,1) = (3,3) = 3(1,1)$$

Vector $v = (1,1)$ is an eigenvector of f

$\lambda = 3$ is the corresponding eigenvalue of v

$7v = (7,7)$ is also an eigenvector of f because

$$f(7,7) = 7f(1,1) = 7 \times 3(1,1) = (21,21) = 3(7,7)$$

In fact, scalar multiple of an eigenvector is also an eigenvector, with the same eigenvalue.

Let A be an $n \times n$ matrix. A vector $v \neq 0$ is called eigenvector of A if

$$Av = \lambda v$$

for some scalar λ. This scalar is called the eigenvalue of A corresponding to eigenvector v.