Lecture 13 (2/4/2019)

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Vector $v \neq 0$ is eigenvector of f if $f(v)$ is parallel to v, i.e.

$f(v) = \lambda v$ for some scalar λ.

* Note: $f(0) = 0$

vector zero in $\mathbb{R}^n$

Why? $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0.$

Ex: If $f(v) = 0$ and $v \neq 0$ then v is an eigenvector of f.

why? $f(v) = 0 v$

eigenvalue

Ex:

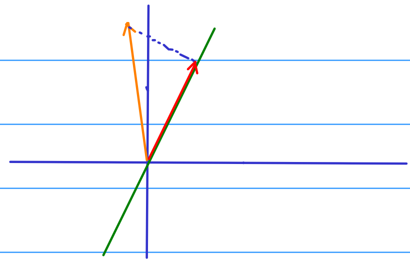
$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$I_2 v = v$ for all $v \in \mathbb{R}^2$

Every nonzero vector in $\mathbb{R}^2$ is an eigenvalue of I_2.

The corresponding eigenvalue is $\lambda = 1$.

Ex: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the projection onto the line $y = 2x$.



There are 2 directions that are preserved under

f: $v_1 = (1,2)$ ---- parallel to the line

$v_2 = (-2,1)$ ---- perpendicular to the line

$f(v_1) = v_1$

eigenvalue $= 1$

$f(v_2) = 0 = 0\, v_2$

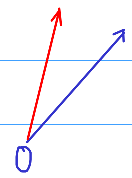
eigenvalue $= 0$

In this case, f has two linearly independent eigenvectors.

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation by $30°$ counterclockwise.

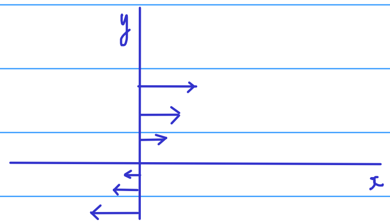
No directions are preserved under f.

f has no real eigenvectors. But it has two complex eigenvectors.



Ex: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x+y, y)$

Only the vectors on the x-axis has direction preserved under f. The only eigenvector is $v = (1,0)$ (and scalar multiples of v).



$$f(v) = v$$

eigenvalue is $\lambda = 1$

* How to compute eigenvectors / eigenvalues of a linear map / matrix?

$$Av = \lambda v \iff \underbrace{(A - \lambda I_n)}_{\substack{n \times n \\ \text{matrix}}} \underbrace{v}_{\substack{\text{nonzero} \\ \text{column vector}}} = 0$$

This equation has two solutions: v and 0.

The coefficient matrix $A - \lambda I_n$ must fail to be invertible.

$$\det(A - \lambda I_n) = 0.$$

Procedure:

- Write matrix $A - \lambda I_n$.
- Compute $\det(A - \lambda I_n)$. This should be a polynomial of degree n.
- Find the roots of this polynomial. These are the eigenvalues of A.
- To each eigenvalue λ, find the corresponding eigenvector v by solving the equation $(A - \lambda I_n)v = 0$.

  This equation should have infinitely many solutions.

$\underline{\underline{Ex}}$:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A - \lambda I_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$\det(A - \lambda I_2) = (1-\lambda)^2 - 4 = (-1-\lambda)(3-\lambda)$

Two roots are $\lambda_1 = -1$ and $\lambda_2 = 3$. These are the two eigenvalues of A.

Find corresponding eigenvectors:

- For $\lambda = -1$:

  We will solve for v from the equation $\underbrace{(A - (-1)I_2)}_{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}} v = 0$.

  Augmented matrix: $\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ (REF)

  this column has no pivot entries

  $x_2 = t$ (free variable)

  $x_1 = -t$ (from the first row)

  The eigenvectors corresponding to eigenvalue $\lambda = -1$ are $(-t, t) = t(-1, 1)$.

- For $\lambda = 3$:

  Do similarly.

$\underline{\underline{Ex}}$:

$$A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2$$

Two complex roots: $\lambda_1 = -1 + i$

$\lambda_2 = -1 - i$

Continue next time.