Lecture 14 (2/11/2019)

Continue the example last time:

$$A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2$$

A has no real-valued eigenvalues, but it has two complex-value eigenvalues : $\lambda_1 = -1+i$

$\lambda_2 = -1-i$

$\lambda_2 = \overline{\lambda_1}$ (complex conjugate)

Eigenvectors corresponding to λ_1:

$$A - \lambda_1 I_2 = A - (-1+i) I_2 = \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix}$$

Recall:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd+(bc-ad)i}{c^2+d^2}$$

$$= \underbrace{\frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i}_{\text{complex number in standard form}}$$

2 ways to transform $A - \lambda I_2$ into REF:

1. Avoid division by complex numbers:

$$[A - \lambda I_2 | 0] = \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix}$$

$$\xrightarrow{R_1 \Leftrightarrow R_2} \begin{bmatrix} 5 & -2-i \\ 2-i & -1 \end{bmatrix}$$

$$\xrightarrow[R_2 = R_2 - (2-i)R_1]{R_1 = R_1/5} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} - \frac{1}{5}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(2)

$$[A-\lambda I_2 \mid 0] = \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right] \xrightarrow{R_1 = R_1/(2-i)} \left[\begin{array}{cc|c} 1 & \frac{-1}{2-i} & 0 \\ 5 & -2-i & 0 \end{array}\right]$$

$$\frac{-1}{2-i} = \frac{-1(2+i)}{2^2+1^2} = \frac{-2}{5} - \frac{i}{5}$$

$$\left[\begin{array}{cc|c} 1 & -2/5 - i/5 & 0 \\ 5 & -2-i & 0 \end{array}\right] \xrightarrow{R_2 - 5R_1} \left[\begin{array}{cc|c} 1 & -2/5 - i/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$ .... free variable

$x_1 + \left(-\frac{2}{5} - \frac{i}{5}\right)x_2 = 0 \implies x_1 = \left(\frac{2}{5} + \frac{i}{5}\right)t$

Eigenvectors corresponding to λ_1: $v = \left(t\left(\frac{2}{5}+\frac{i}{5}\right), t\right)$

$= t\left(\frac{2}{5} + \frac{i}{5}, 1\right)$, t is arbitrary complex number

Pick $t=5$: $v_1 = (2+i, 5)$

One does similarly to obtain eigenvectors of λ_2. Alternatively, one can use the observation:

$$Av_1 = \lambda_1 v_1$$

Take the complex conjugate of both sides:

$$\underbrace{\bar{A}}_{\text{matrix whose entries are complex conjugate of entries of } A} \bar{v}_1 = \bar{\lambda}_1 \bar{v}_1$$

$$\left.\begin{array}{l} \bar{A} = A \\ \bar{\lambda}_1 = \lambda_2 \end{array}\right\} \implies A\bar{v}_1 = \lambda_2 \bar{v}_1$$

$\bar{v}_1 = (\overline{2+i}, \bar{5}) = (2-i, 5)$ is an eigenvector corresponding to λ_2.

Conclusion:

A has two complex eigenvectors: $v_1 = (2+i, 5)$ ...... $\lambda_1 = -1+i$

$v_2 = (2-i, 5)$ ...... $\lambda_2 = -1-i$

* Nonlinear function:

How to compute $e^{0.1}$ using $\underbrace{\text{addition, subtraction, multiplication and division}}_{\text{these are the operation one can do by hand}}$?

These operations allow one to evaluate values of polynomials.

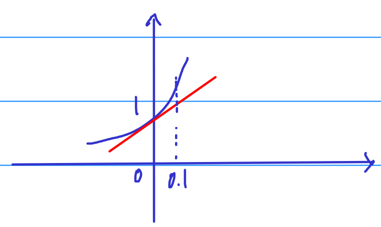
Ex: $f(x) = 2x^3 - 2x^2 + 3x + 1$

$f(0.1) = 2(0.1)(0.1)(0.1) - 2(0.1)(0.1) + 3(0.1) + 1 = \ldots$

* Idea: approximate a function by polynomial before evaluating.

$0.1 \approx 0$, so $e^{0.1} \approx e^0 = 1$

How to get a better approximation?



$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{eq. of tangent line}}$

$e^{0.1} \approx e^0 + e^0(0.1 - 0) = 1.1$

The approximation $f(x) \approx f(x_0)$ becomes exact when f is a constant function

" $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ " $\underbrace{\text{polynomial of degree} \leq 1}_{f(x) = ax+b}$