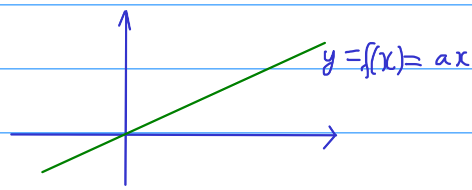
Lecture 15 (2/13/2019)

How to compute $f(x)$ knowing f at x_0?



If $f: \mathbb{R} \to \mathbb{R}$ is linear, one only needs to know $f(x_0)$. With $a = \frac{f(x_0)}{x_0}$, the function f is completely determined.

What if f is nonlinear? What do we need to know about f at x_0 in order to evaluate $f(x)$ for any x?

Naive approximation: $f(x) \approx f(x_0)$ ...... when no information about f at x_0 is available, except for $f(x_0)$.

Better appr. : $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

for this appr. to work, one needs to know both $f(x_0)$ and $f'(x_0)$.

If we know more derivatives of f at x_0, we expect to get better appro.

Suppose $f(x_0), f'(x_0), f''(x_0)$ are known.

$$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{g(x)} + \underbrace{\alpha (x - x_0)^2}_{2^{nd} \text{ order correction term}}$$

Put $h(x) = f(x) - g(x)$. Then

$$\lim_{x \to x_0} \frac{h(x)}{(x - x_0)^2} \overset{\text{L'Hospital}}{=\!=} \lim_{x \to x_0} \frac{h'(x)}{2(x - x_0)} \overset{\text{L'Hospital}}{=\!=} \lim_{x \to x_0} \frac{h''(x)}{2}$$

$$= \frac{f''(x_0)}{2}$$

Choose $\alpha = \frac{f''(x_0)}{2}$.

Approximation	Exact when .....
$f(x) \approx f(x_0)$	f is constant: $f(x) = a$ for all x.
$f(x) \approx f(x_0) + \underbrace{f'(x_0)(x-x_0)}_{\text{1st order correction term}}$	$f(x) = ax+b$
$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \underbrace{\frac{f''(x_0)}{2}(x-x_0)^2}_{\text{2nd order correction term}}$	$f(x) = ax^2+bx+c$
.....	.....
$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{T_n(x)}$	$f(x)$ is a polynomial of degree n

$$T_n(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is called n'th (order) Taylor polynomial of f about x_0.

When $x_0 = 0$:

$$T_n(x) = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

is also called Maclaurin polynomial.

Applications:

* How to compute $\sqrt{2}$ by hand?

$f(x) = \sqrt{x}$

Want to evaluate $f(2)$ knowing f at 1:

$$f(x) = x^{1/2} \quad \cdots\cdots \quad f(1) = 1$$

$$f'(x) = \frac{1}{2} x^{-1/2} \quad \cdots\cdots \quad f'(1) = \frac{1}{2}$$

$$f''(x) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) x^{-3/2} \quad \cdots\cdots \quad f''(1) = -\frac{1}{4}$$

$$f'''(x) = \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) x^{-3/2} \quad \cdots\cdots \quad f''(1) = \frac{3}{8}$$

$$\cdots\cdots$$

The Taylor polynomials of f about 1 are:

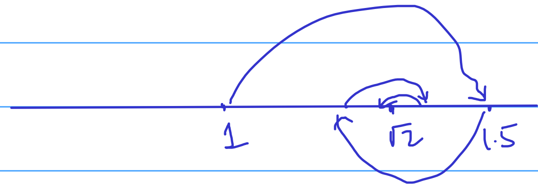
$$T_0(x) = f(1) = 1$$

$$T_1(x) = 1 + \underbrace{f'(1)(x-1)}_{1^{st}\text{ order corr. term}} = 1 + \frac{1}{2}(x-1)$$

$$T_2(x) = 1 + \frac{1}{2}(x-1) + \underbrace{\frac{f''(1)}{2}(x-1)^2}_{2^{nd}\text{ order corr. term}} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

$$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \underbrace{\frac{f'''(1)}{6}(x-1)^3}_{3^{rd}\text{ order corr. term}} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

Order of appro.	Appro. of $\sqrt{2}$
0	$\sqrt{2} \approx T_0(2) = 1$
1	$\sqrt{2} \approx T_1(2) = 1 + 0.5 = 1.5$
2	$\sqrt{2} \approx T_2(2) = 1.5 - \frac{1}{8} = 1.375$
3	$\sqrt{2} \approx T_3(2) = 1.375 + \frac{1}{16} = 1.4375$



* How to evaluate the definite integral

$$\int_0^1 \cos(x^2)\,dx \quad ?$$

The antiderivative of $f(x) = \cos(x^2)$ is not an elementary function. Elementary function is a function obtained from the

- polynomials,
- exponential, logorithm functions,
- trigonometric functions, inverse trig. functions,

by combining them using addition, subtraction, multiplication, division, composition.

It's not clear how to use the Fundamental Theorem of Calculus to compute this definite integral.

In practice, one only needs to find a good approximation of $\int_0^1 \cos(x^2)\,dx$ and, if possible, know how to control (reduce) the error.

$$f(0) = \cos(0) = 1$$

$$f'(x) = -2x\sin(x^2) \implies f'(0) = 0$$

$$f''(x) = -2\sin(x^2) - 4x^2\cos(x^2) \implies f''(0) = 0$$

$$f'''(x) = \cdots\cdots$$

One can compute (by hand) $f(0), f'(0), f''(0), f^{(3)}(0), \ldots$

$$f(x) \approx T_n(x)$$

n ← order of approximation

$$\int_0^1 \cos(x^2)\,dx \approx \int_0^1 T_n(x)\,dx$$

One knows how to integrate a polynomial function.