Lecture 16 (2/15/2019)

Recall: $f(x) \approx T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= \underbrace{f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{n\text{'th order Taylor poly. of } f \text{ about } a}$$

Ex:

$f(x) = \ln x \quad , \quad x > 0$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = (-1)\, x^{-2}$

$f'''(x) = (-2)(-1)\, x^{-3}$

$f^{(4)}(x) = (-3)(-2)(-1)\, x^{-4}$

.....

$f^{(k)}(x) = (-(k-1)) \cdots (-2)(-1)\, x^{-k} = (-1)^{k-1}(k-1)!\, x^{-k}$

$f^{(k)}(1) = (-1)^{k-1}(k-1)!$

$\frac{f^{(k)}(1)}{k!} = \frac{(-1)^{k-1}}{k}$

Taylor polynomials about base point $a = 1$:

$T_0(x) = f(1) = 0$

$$T_1(x) = 0 + \underbrace{\frac{f'(1)}{1!}(x-1)}_{1^{st} \text{ correction term}} = (x-1)$$

$$T_2(x) = (x-1) + \underbrace{\frac{f''(1)}{2!}(x-1)^2}_{2^{nd} \text{ correction term}} = (x-1) - \frac{1}{2}(x-1)^2$$

$$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

......

$$T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots + \frac{(-1)^{n-1}}{n}(x-1)^n$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$$

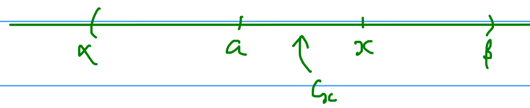
How to control the error $R_n(x) = f(x) - T_n(x)$?

<u>Theorem:</u> (by Lagrange)

If f is differentiable $(n+1)$ times on an interval $\underbrace{(\alpha, \beta)}_{\text{containing } a}$ then

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad (x \in (\alpha, \beta))$$

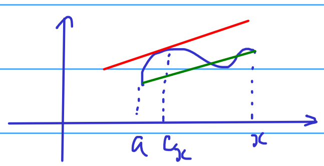
for some c lying between a and x, depending on n and x.



This is a generalization of the mean-value theorem (for higher-order derivatives). Why?

Consider $n = 0$ (the simplest case):

$$f(x) - T_0(x) = f(x) - f(a) = f'(c_x)(x-a)$$



<u>Observation:</u> what does it take to make the error term $R_n(x)$ small?

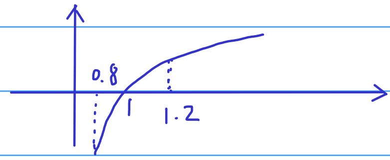
Suppose the $(n+1)$'st derivative of f is bounded, i.e.

$$|f^{(n+1)}(x)| \leq M \quad \text{for all } x \in (\alpha, \beta)$$

then $R_n(x)$ is small if x is close to a or n is large.

How to estimate the error term of the logarithm function?

$f(x) = \ln x, \qquad x \in [0.8, 1.2]$



$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1}$$

$$= \frac{(-1)^n c^{-n-1}}{n+1}(x-1)^{n+1}$$

$c \in [0.8, 1.2]$

$$|R_n(x)| = \frac{|c|^{-(n+1)}}{n+1}|x-1|^{n+1} \leq \frac{(\frac{5}{4})^{n+1}}{n+1}\left(\frac{1}{5}\right)^{n+1} = \frac{1}{n+1}\left(\frac{1}{4}\right)^{n+1}$$

What is the minimum value of n such that the error doesn't exceed 10^{-4} for any $x \in [0.8, 1.2]$?

$$\underbrace{\frac{1}{n+1}\left(\frac{1}{4}\right)^{n+1}} < 10^{-4}$$

doesn't exceed $\left(\frac{1}{4}\right)^{n+1} = 4^{-(n+1)}$

Any value of n such that $4^{-(n+1)} < 10^{-4}$ would do it.

Want: $\underbrace{\ln 4^{-(n+1)}}_{-(n+1)\ln 4} < \underbrace{\ln 10^{-3}}_{-4 \ln 10}$

Divide both sides by $-\ln 4$:

$$n+1 > \frac{4 \ln 10}{\ln 4} \approx 6.64$$

$n = 6$ would do it.