Lecture 17 (21/8/2019)

$$f(x) = \underbrace{T_n(x)}_{\text{n'th Taylor poly.}} + \underbrace{R_n(x)}_{\text{error term}}$$

* Lagrange's theorem:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

for some c between x and x_0.

Recall:

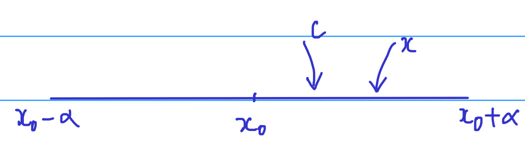
$$\underbrace{\ln x}_{f(x)} \approx \underbrace{0 + \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n}}_{T_n(x)}$$

when $x=3$, $T_n(3)$ is not a good approximation for $\ln 3$ because the $(n+1)$'st correction term is

$$(-1)^n \frac{(3-1)^{n+1}}{n+1} = (-1)^n \frac{2^{n+1}}{n+1}$$

doesn't tend to zero as $n \to \infty$.

Assume: $|f^{(n+1)}(x)| \leq M$ for all $x \in (x_0-\alpha, x_0+\alpha)$ and for all n.



$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

$$|R_n(x)| \leq \frac{M}{(n+1)!}\alpha^{n+1}$$

For fixed n, $R_n(x) \to 0$ as $x \to x_0$.

Approximation gets better when x is close to x_0.

For fixed α, $R_n(x) \to 0$ as $n \to \infty$.

why? the quotient

$$\frac{\frac{M}{(n+1)!}\alpha^{n+1}}{\frac{M}{n!}\alpha^{n}} = \frac{\alpha}{n+1} < \frac{1}{2} \quad \text{for } n \text{ large}$$

$f(x)$ can be approximated using $f(x_0), f'(x_0), f''(x_0), \dots, f^{(n)}(x_0)$. when x is further from x_0, n is required to be bigger in order to keep the error term small.

* Observation: $f(x) = T_n(x) + R_n(x) = \lim_{n\to\infty} T_n(x)$

Formally,

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots}_{\text{an infinite sum, called Taylor series}}$$

How to | define / make sense of | an infinite sum?

The sum $a_1 + a_2 + a_3 + \dots$ is called an infinite sum, or series.

The sum $S_n = a_1 + a_2 + \dots + a_n$ is called a partial sum.

Def:

If $\lim S_n = S$ exists then the series $\sum_{n=1}^{\infty} a_n$ is said to converge and $\sum_{n=1}^{\infty} a_n = \underbrace{S}_{\text{sum of the series}}$

If $\lim S_n = \pm\infty$ or doesn't exist then the series is said to diverge.

Ex:

Consider the infinite sum $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$

Partial sum: $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$

How to compute S_n?

$$2S_n = 2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$- \quad S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}$$

$$S_n = 2 - \frac{1}{2^n}$$

Because $\lim S_n = 2$, the series converges to 2.

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = 2$$

Ex: Consider the series $1 + q + q^2 + q^3 + q^4 + \cdots$

Partial sum $S_n = 1 + q + q^2 + \cdots + q^n$

How to compute S_n?

$$qS_n = q + q^2 + q^3 + \cdots + q^n + q^{n+1}$$

$$- \quad S_n = 1 + q + q^2 + q^3 + \cdots + q^n$$

$$(q-1)S_n = q^{n+1} - 1$$

$$S_n = \frac{q^{n+1} - 1}{q - 1}$$

This sequence converges to $\frac{-1}{q-1}$ if $-1 < q < 1$.

It doesn't converge if $q \leq -1$ or $q \geq 1$.

$$1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q} \quad \text{for all } q \in (-1, 1).$$